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A Realist Interpretation Of The Causal-inertial Structure Of Spacetime

Herbert Korte

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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS REÇUE

A REALIST INTERPRETATION
OF THE CAUSAL-INERTIAL STRUCTURE
OF SPACETIME

by

Herbert Korte

Department of Philosophy

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of the requirements for the degree of
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ABSTRACT

The central aim of this dissertation is to clarify, defend and develop a realist field ontology of the causal-inertial structure of spacetime forcefully advanced by Hermann Weyl. Weyl's field ontology of spacetime structure may roughly be described as follows. The Special and General as well as the non-relativistic spacetime theories are principle theories of spacetime structure. They all postulate various structural constraints, and events within spacetime are held to satisfy these constraints. When interpreted physically, these mathematical structures correspond to physical structural fields on spacetime.

The thesis begins with a discussion of the significance of Riemann's Dynamical Hypothesis within the context of the debate between geometrical realism and conventionalism. In particular, Grünbaum's unyielding insistance that Riemann's work must be seen as the immediate progenitor of his brand of geometrical conventionalism is challenged and shown to be unsupported by a textual analysis of the relevant works by Riemann. Moreover, it is shown that Grünbaum's interpretation is plainly at odds with the basic intentions of Riemann when one takes into account some aspects of the historical development of differential geometry. The first two chapters provide a historical/exegetical analysis of Riemann's Habilitationsvortrag. Chapter 3 presents a general account of geometrical conventionalism apart from

issues of its alleged historical ancestry as seen by Grünbaum. This is followed by a critical examination of Grünbaum's interpretation of Riemann.

Grünbaum has repeatedly argued that Weyl's interpretation of Riemann accords with his. It is shown, however, that Weyl's understanding of Riemann is the direct opposite of that of Grünbaum. Chapter 4 provides an examination of Weyl's philosophy of geometry and his interpretation of Riemann. It is shown that Weyl considers Riemann's work to be essentially progenitorial of his version of geometrical realism.

In Chapter 5 an account of the causal-inertial method first introduced by Weyl (1921) and later developed by Ehlers, Pirani and Schild (1972), is provided. The 4-dimensional pseudo-Riemannian manifold is the mathematical model of the physical spacetime of General Relativity. Weyl distinguished between two primitive structures of the model, viz., the <u>conformal</u> and <u>projective</u> structures. Weyl suggested that the conformal structure represents the <u>causal field</u> governing light propagation and the projective structure represents the inertial or <u>guiding field</u> governing all free (fall) motions. Using these structures and their compatibility relation, Ehlers, Pirani and Schild have constructed an improved version of Weyl's causal-inertial method and derived a unique pseudo-Riemannian metric solely as a consequence of constructive axioms concerning light propagation and free (fall) motion.

The geodesic method has recently fallen under criticism. One has argued that any criteria that determine which bodies are in free (fall) motion presuppose the specification of spacetime structures beyond those implied by the local differential topological structure. Since geometrical conventionalism regards the world to be factually definite only up to its local differential topological descriptions these structures are considered to be conventional. These criticisms are discussed in Chapter 6 and are shown to be unfounded. It is shown that the causal-inertial method is not beset with a circularity problem and that the construction of a unique affine and pseudo-Riemannian structure, from a few qualitative assumptions concerning causal propagation and free (fall) motion constitutes a convention-free -- and in relevant respects -- theory-independent body of evidence that can adjudicate between spacetime geometries and hence between spacetime theories that postulate them. First, quite general path structures are discussed which are not defined at the outset in terms of geodesic paths and which require for their description only the local differential topological structure. Secondly, a number of theorems are discussed that serve as purely local differential topological criteria for singling out from among the general path structures the geodesic ones that represent the inertial structure of spacetime. It is shown, therefore, that we do have epistemic access to light propagation and free (fall) motion in a way that does not beset the causal-inertial method with either logical or derivatively epistemological circularity.

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CHAPTER 1

THE DEVELOPMENT OF THE MANIFOLD CONCEPT

1.1 SOME BIOGRAPHICAL REMARKS CONCERNING RIEMANN'S INAUGURAL LECTURE OF 1854

It was on June 10, 1854, that the faculty of the University of Göttingen heard Riemann's inaugural lecture entitled "Über die Hypothesen, welche der Geometrie zu Grunde Liegen" (On the Hypotheses which lie at the Foundation of Geometry)[1]. Although the ideas contained within it proved to be the most influential in the entire history of differential geometry, the lecture was not published until 1868, two years after Riemann's death, by his friend Dedekind.

We can be fairly certain that Riemann would have preferred to revise his lecture considerably before publishing it. Considering the wealth and depth of ideas contained in it, the time for its preparation available to Riemann certainly seems remarkably short. In his biography of Riemann, Dedekind tells us that Riemann had completed his Habilitationsschrift ("Ueber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe") by the end of 1853[2]. However, the theme and date of his Habilitationsvortrag (trial lecture) still had to be arranged. When submitting his Habilitationsschrift to the faculty, Riemann suggested three themes in the following order of preference:

(1) Geschichte der Frage über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe. (History of the question concerning the representation of a function

by a trigonometric series.)

(2) Über die Auflösung zweiter Gleichungen zweiten Grades mit zwei unbekannten Grössen. (About the solution of two equations of second degree in two unknowns.)

(3) Über die Hypothesen welche der Geometrie zu Grunde liegen. (About the Hypotheses which lie at the basis of Geometry.)[3]

At the time of submission Riemann had the first two proposals fully prepared and hoped, of course, that Gauss would choose one of them. But Gauss, who usually honored the candidate's preference, decided because of his keen interest in the subject, to examine the young Riemann on the third proposal instead. Consequently, Riemann found himself pressed for time. Concerning this he wrote to his brother Wilhelm on 28 December 1853:

> Mit meinen Arbeiten steht es jetzt so ziemlich; ich habe Anfangs December meine Habilitationsschrift abgeliefert und musste dabei drei Themata zur Probelesung vorschlagen, von denen dann die Facultät eines wählt. Die beiden ersten hatte ich fertig und hoffte, dass man eins davon nehmen würde; Gauss aber hatte das dritte gewählt, und so bin ich nun wieder etwas in der Klemme, da ich dies noch ausarbeiten muss.[4]

> (My work is going reasonably well; at the beginning of December I submitted my Habilitationsvortrag and I had to propose at the same time three themes for the trial lecture, from which the faculty will choose one. I had the first two complete and hoped that they would choose one of them; Gauss, however, chose the third and I am therefore now in something of a difficulty, since I must still work this one out.)[5]

Riemann's difficulty was aggravated by his failing health in the winter of 1854. It was not until two weeks after Easter that Riemann was well enough to prepare (in full earnest) for his trial lecture.

Riemann still faced a further difficulty. It was a well-established custom that an inaugural lecture took place before the entire faculty. This meant that Riemann was required to communicate inherently abstract and difficult mathematical ideas to an audience largely made up of non-mathematicians.

Gauss, of course, wanted to be present at Riemann's lecture. But his own health was so poor, however, that he did not immediately decide on the lecture date. It therefore came as a surprise to Riemann when Gauss all of a sudden set the date for 10 June 1854. In a letter to his brother of 26 June 1854 Riemann wrote:

> Da entschloss er sich plötzlich auf mein wiederholtes Bitten, „um die Sache vom Halse los zu werden", am Freitag nach Pfingsten Mittag dass Colloquium auf den anderen Tag um halb elf anzusetzen und so war ich am Sonnabend um eins glücklich damit fertig.[6]
>
> (Then suddenly, after my repeated requests 'to take the yoke from my neck' he decided at noon on the Friday after Whitsun to set the colloquium for the next day at 10:30 and so on Saturday at one o'clock I was happily finished with it.)

We are told by Dedekind that Gauss sat at the lecture "which surpassed all his expectations, and on the way back from the faculty meeting he spoke to Wilhelm Weber with the greatest appreciation, and with an excitement rare for him, about the depth of the ideas presented by Riemann."[7]

1.2 THE MAIN THRUST OF RIEMANN'S IDEAS

Broadly speaking, one of the significant contributions in Riemann's lecture that is particularly relevant for the epistemology of geometry consists in an elucidation of the logical relationship between geometry viewed as a branch of pure mathematics and its application to physical space. In the introduction ("Plan of the Investigation") Riemann points out that the concept of space, and the basic concepts for the construction of the geometric properties of space, had been assumed by geometers without any clear notion as to the relationship between these two presuppositions. Riemann attributes the confusion surrounding the status of non-Euclidean geometry to an insufficient grasp of the general concept of a "multiply extended magnitude". It is this lack of understanding, according to Riemann, which prevented geometers from separating (to use modern terminology) the topological features of space from its metrical features. The first task Riemann sets himself is to work out in Part I of the lecture the general concept of a "multiply extended magnitude". In the introduction he says:

> Bekanntlich setzt die Geometrie sowohl den Begriff des Raumes, als die ersten Grundbegriffe für die Constructionen im Raume als etwas Gegebenes voraus. Sie giebt von ihnen nur Nominaldefinitionen, während die wesentlichen Bestimmungen in Form von Axiomen auftreten. Das Verhältnis dieser Voraussetzungen bleibt dabei im Dunkeln; man sieht weder ein, ob und in wie weit ihre Verbindung nothwendig, noch a priori, ob sie möglich ist.
>
> Diese Dunkelheit wurde auch von Euklid bis auf Legendre, um den berühmtesten neueren Bearbeiter der Geometrie zu nennen,

weder von den Mathematikern, noch von den Philosophen, welche sich damit beschäftigten, gehoben. Es hat dieses seinen Grund wohl darin, dass der allgemeine Begriff mehrfach ausgedehnter Grössen, unter welchem die Raumgrössen enthalten sind, ganz unbearbeitet blieb. Ich habe mir daher zunächst die Aufgabe gestellt, den Begriff einer mehrfach ausgedehnten Grösse aus allgemeinen Grössenbegriffen zu construiren.[8]

(As is well known, geometry presupposes the concept of space as well as the first fundamental concepts for constructions in space, as given in advance. It gives only nominal definitions for them, while their essential specifications appear in the form of axioms. The relationship between these presuppositions [the concept of space and the basic geometrical properties of space, H. K.] is left in the dark; one does not recognize whether and to what extent their connection is necessary nor does one see a priori whether any connection between them is even possible.

From Euclid to Legendre, to name the most renowned of the modern researchers on geometry, this darkness has been dispelled neither by mathematicians nor by philosophers who have concerned themselves with it. This is undoubtedly, because the general concept of multiply extended magnitudes which includes spatial magnitudes, remains completely unexplored. I have therefore first set myself the task of constructing the concept of a multiply extended magnitude from general concepts of magnitude.)

Riemann thought that a clear understanding of the general concept of a "multiply extended magnitude" is a necessary prerequisite for a proper understanding of the precise relationship between the general concept of space on the one hand, and the spatial geometrical attributes on the other. In particular, this will show that the concept of a "multiply extended magnitude" -- of which space is just a special instance -- does not itself involve the specification of metric relationships. Rather, the mathematical representation of a continuous triply extended magnitude, such as physical space, admits of different metrical structures. Consequently, the theorems of physical geometry cannot be deduced from the general concept of a 'triply extended magnitude' (in modern terminology, a three dimensional continuous topological space). Riemann says:

> Es wird daraus hervorgehen, dass eine mehrfach ausgedehnte Grösse verschiedener Massverhältnisse fähig ist und der Raum also nur einen besonderen Fall einer dreifach ausgedehnten Grösse bildet. Hiervon aber ist eine nothwendige Folge, dass die Sätze der Geometrie sich nicht aus allgemeinen Grössenbegriffen ableiten lassen, sondern dass diejenigen Eigenschaffen, durch welche sich der Raum von anderen denkbaren dreifach ausgedehnten Grössen unterscheidet, nur aus der Erfahrung entnommen werden könne.[4]
>
> (What will emerge from this, is, that a multiply extended magnitude is capable of various metric relations, and that [physical, H. K.] space constitutes only a special case of a triply extended magnitude. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of magnitude, but that those properties which distinguish [physical, H. K.] space from other conceivable triply extended magnitudes can only be discovered through experience.)

The discussion on n-fold magnitudes in Part I contains an informal theory of topological manifolds and dimensions. Riemann is quite clear about this. In the table of contents he indicates with a footnote, that the discussion on n-fold magnitudes is intended as a preliminary work for contributions to analysis situs.

> Art. I bilden zugleich die Vorarbeit für

Beitrage zur analysis situs.[10]

(Part I constitutes at the same time the preliminary work for contributions to analysis situs.)

In his effort to develop the concept of a "multiply extended magnitude", Riemann employs the notion of a manifold (Mannigfaltigkeit). He considered the latter to be more general than the former, and was attempting to express through the notion of a manifold something like our modern concept of a topological manifold. Hence, when Riemann speaks of a "general concept of a multiply extended magnitude" he may have been thinking about something like our modern notion of a "general topological space".

Riemann's work on the general notion of a manifold remains however intimately tied to his specialized concerns with algebraic function theory.[11] His earlier investigations in complex function theory as presented in his Dissertation "Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse"[12], led him to study the topology of surfaces, that is, two dimensional manifolds. It was therefore quite natural for him to investigate in his lecture the topology of n-dimensional hypersurfaces. After he first distinguishes between discrete and continuous manifolds, Riemann describes how measurement in a continuous space involves the superposition of some "congruence standard" under transport and that in the absence of such a standard "one can compare two magnitudes only when one is part of the other, and then only as to 'more' or ''less'; not as to 'how much'."[13] Since immediately following that remark Riemann refers to his own way of dealing with multi-valued functions by means of Riemann surfaces, he clearly wishes to emphasize that a topological treatment of manifolds is important and necessary for the development and understanding of some branches of mathematics.[14] Riemann says:

> Die Untersuchungen, welche sich in diesen Falle über sie austellen lassen, bilden einen allgemeinen von Massbestimmungen unabhängig von der Lage existirend und nicht als durch eine Einheit ausdrückbar, sondern als Gebiete in einer Mannigfaltigkeit betrachtet werden. Solche Untersuchungen sind für mehrere Theile der Mathematik, namentlich für die Behandlung der mehrwerthigen analytischen Functionen ein Bedürfniss geworden, und der Mangel derselben ist wohl eine Hauptsache, dass der berühmte Abel'sche Satz und die Leistungen von Lagrange, Pfaff, Jacobi für die allgemeine Theorie der Differentialgleichungen so lange unfruchtbar geblieben sind. Für den gegenwärtigen Zweck genügt es, aus diesem allgemeinen Theile der Lehre von den ausgedehnten Grössen, wo weiter nichts vorausgesetzt wird als was in dem Begriffe derselben schon enthalten ist, zwei Punkte hervorzuheben, wovon der erste die Erzeugung des Begriffs einer mehrfach ausgedehnten Mannigfaltigkeit, der zweite die Zurückführung der Ortsbestimmungen in einer gegebenen Mannigfaltigkeit auf Quantitätsbestimmungen betrifft und das wesentliche Kennzeichen einer n-fachen Ausdehnung deutlich machen wird.[15]

> (The investigations which can be carried out in this case form a general division of the science of magnitude, independent of measurement, where magnitudes are regarded, not as existing independent of position and not as expressible in terms of a unit, but as regions in a manifold. Such investigations have become a necessity for several parts of mathematics, eg., for the treatment of many-valued analytic func-

tions, and the lack of such studies is one of the principal reasons why the celebrated theorem of Abel and the contributions of Lagrange, Pfaff and Jacobi to the general theory of differential equations have remained unfruitful for so long. From this part of the science of extended quantity, a part which proceeds without any further assumptions, it suffices for the present purposes to emphasize two points, which will make clear the essential nature of an n-fold extension. The first of these concerns the generation of the concept of a multiply extended magnitude, the second involves reducing position-fixing in a given manifold to metrical determinations.)

1.3 THE INFLUENCE OF GAUSS AND GRASSMANN

Apart from his own interest and earlier researches in the theory of complex and algebraic functions, it was the broader context of Gauss' mathematical work on higher dimensional surfaces and Hermann Grassmann's Ausdehnungslehre which motivated and influenced Riemann's thinking about the general question of the logical relationship between geometry viewed as a branch of pure mathematics and its application to physical space.

It was in large measure due to the new attitude towards geometry expressed by Gauss and Grassmann, that a new philosophical viewpoint emerged during the first part of the nineteenth century. Generally speaking, almost all mathematicians and philosophers prior to Gauss and Grassmann considered the study of geometry to be essentially a science of three dimensional physical space.

For Gauss and Grassmann the study of geometry became freed from concerns with physical space and developed into a branch of pure mathematics. Spatial intuition and images were considered to be of heuristic value only and could in the final analysis be eliminated from geometrical theories.

The construction of hyperbolic non-Euclidean geometry by Gauss, Lobatschevsky and Bolyai at the beginning of the nineteenth century demanded that the claims of the preceding century for the absolute truth of Euclidean geometry be abandoned. No longer could an unlimited confidence be attributed to classical geometrical intuition. According to Gauss the study of abstract geometry is concerned with n-dimensional manifolds in general, whereas the geometry of physical space is restricted to the study of a specific three dimensional manifold. Gauss associated abstract geometry with geometrical situs, or what we would call topology, and was deeply interested in the geometry of hyperspaces during the first four decades of the nineteenth century.

Grassmann conceived of mathematics as a science of pure forms of thought, a "formal science" as he called it, that is logically independent of experience. In his Die lineale Ausdehnungslehre (1844)[16] he develops a theory of extension (Ausdehnung) which does not rely on spatial intuition. Freed from three-dimensional constraints of physical space, it is pure mathematics with continuous quantities as its subject matter. Grassmann's Ausdehnungslehre, or abstract geometry, is not only prior to our knowledge of physical space, but is even prior to number and arithmetic as such. It does not presuppose the latter; rather numbers, arithmetical relations and geometrical properties are derivable from continuous quantities. Although Grassmann followed Leibniz in regarding the basis of his "formal science" as definitions and not as axioms, what he had in mind was nonetheless something like Axiomatics in the modern sense.

The emergence of non-Euclidean geometry helped to undermine the trustworthiness of empirical geometrical intuition and to motivate developments toward the axiomatic approach in general. This meant that the shift in attitude towards geometry that was first initiated by Gauss and Grassmann gave rise not only to philosophical questions about the logical relation between pure and applied geometry, but it also caused a gradual revision in attitude toward the nature of mathematical objects and mathematical truth.

According to Euclid and the classical Greeks physical space was considered to be something completely rational. The decisions concerning the truth or falsity of geometric propositions were to be made with reference to it. During the Renaissance period mathematics embraced many sciences and the success of its applications to such fields as navigation, the mechanical arts and natural philosophy was the main reason for the confidence which it inspired.

Because of their notion of n-dimensional manifolds, as well as their understanding of non-Euclidean geometry, Gauss, Grassmann and Riemann clearly saw the subject of geometry as encompassing much more than the study of geometrical relations deducible from physical space. The purely logical treatment of geometry, the complete abandonment of all appeal to empirical intuition was not developed, however, until Pasch, who first formulated it as a program and carried it out with complete rigour.[17] Subsequent to Pasch's work, there appeared various axiomatic treatises emulating his approach. The most famous of which was David Hilbert's <u>Grundlagen der Geometrie</u>[18] which appeared in 1899 and became the model of modern Axiomatics.

The conception that geometrical facts are ultimately independent of their meaning or sense and consist solely in terms of their logical relations was pushed by Hilbert to its logical consequence by emphasizing that even the names of the basic concepts of a mathematical system may be chosen arbitrarily. Thus Hilbert showed that in the subject of geometry, a branch of mathematics which before Gauss, Grassmann and Riemann's time had been considered as one of the closest to external reality, the mathematician has full freedom in the choice of his postulates.[19] In 1902 Poincare insisted that the axioms of geometry are pure conventions and that the ordinary notion of 'truth' does not apply to them. Mathematical truth came to be regarded as consisting entirely in logical deductions from premises put forward arbitrarily as axioms. The latter are "neither synthetic a priori intuitions nor experimental facts. They are conventions" and "are only definitions in disguise".[20]

1.4 RIEMANN'S CONCEPTUAL SEPARATION THESIS

Prior then to the works of Gauss, Grassmann and Riemann the study of geometry was limited to empirical intuitions and images of the three dimensional physical space. The latter was thought of as having definite metrical attributes. The task of the geometer was to take bodies in that space and work with them. Under the influence of Gauss and Grassman, Riemann's great philosophical contribution consisted in the demonstration that, unlike the case of a discrete manifold, where the determination of a set necessarily implies the determination of its quantity or cardinal number, in the case of a continuous manifold, the concept of such a manifold and of its continuity properties, can be separated from its metrical structure.

It is in the very beginning of Section I (Part I) where Riemann introduces what I shall refer to as <u>Riemann's Conceptual Separation Thesis</u>:

> Grössenbegriffe sind nur da möglich, wo sich ein allgemeiner Begriff vorfindet, der verschiedene Bestimmungsweisen zulässt. Je nachdem unter diesen Bestimmungweisen von einer zu einer andern ein stetiger Übergang stattfindet oder nicht, bilden sie eine stetige oder discrete Mannigfaltigkeit; die einzelnen Bestimmungsweisen heissen im ersten Falle Punkte, im letztern Elemente dieser Mannigfaltigkeit.
>
> . . .
>
> Bestimmte, durch ein Merkmal oder eine Grenze unterschiedene Theile einer Mannigfaltigkeit heissen Quanta. Ihre Vergleichung der Quantität nach geschieht bei den discreten Grössen durch Zählung, bei den stätigen durch Messung. Das Messen besteht in einem Aufeinanderlegen der zu vergleichen-

den Grössen; zum Messen wird also ein Mittel erfordert, die eine Grösse als Massstab für die andere fortzutragen. Fehlt dieses, so kann man zwei Grössen nur vergleichen, wenn die eine ein Theil der andern ist, und auch dann nur das Mehr oder Minder, nicht das Wieviel entscheiden. Die Untersuchungen, welche sich in diesem Falle über sie anstellen lassen, bilden einen allgemeinen von Massbestimmungen unabhängigen Theil der Grössenlehre, wo die Grössen nicht als unabhängig von der Lage existirend und nicht als durch eine Einheit ausdrückbar, sondern als Gebiete in einer Mannigfaltigkeit betrachtet werden.[21]

(Notions of magnitude are possible only when there already exists a general concept which admits particular instances. These instances form either a continuous or a discrete manifold, depending on whether or not a continuous transition between any two instances can be found. Individual instances are called points in the first case and elements of the manifold in the second.

. . .

Particular parts of a manifold, distinguished by a mark or by a boundary, are called quanta. Their quantitative comparison is effected in the case of discrete magnitudes by counting, in the case of continuous magnitudes by measurement. Measuring involves the superposition of the magnitudes to be compared; it therefore requires a means of transporting one magnitude to be used as a standard for the other. Otherwise, one can compare two magnitudes only when one is a part of the other, and then only as to "more" or "less", not as to "how much". The investigations which can be carried out in this case form a general division of the science of magnitude, independent of measurement, where magnitudes are regarded not as existing independent of position and not as expressible in terms of a unit, but as regions in a manifold.)

It is clear that in the second part of the above citation Riemann is referring to topological manifolds when he says "The investigations which can be carried out in this case form a general division of the science of magnitude, independent of measurement, where magnitudes are regarded, not as existing independent of position and not as expressible in terms of a unit, but as regions in a manifold." This is particularly emphasized by the fact that immediately after that remark he mentions multivalued analytic functions and issues related to complex analysis.

It should also be borne in mind that the term "Grösse" used by Riemann in this context does not refer to measured quantities but refers to the general notion of extension that does not necessarily involve a metrical structure.[22] Perhaps Riemann could have used Grassmann's term "Ausdehnung" instead of "Grösse".

The second part of the above citation constitutes Riemann's main statement in his lecture on the science of analysis situs. It is remarkably similar to another statement of his conception of topology which occurs in one of the papers of a set of papers known as "Theorie der Abelschen Funktion" published three years after the inaugural address:

> Bei der Untersuchung der Funktionen, welche aus der Integration vollständiger Differentialen entstehen, sind einige der analysis situs angehörige Sätze fast unentbehrlich. Mit diesem von Leibniz, wenn auch vielleicht nicht ganz in derselben Bedeutung, gebrauchten Namen darf wohl ein Theil der Lehre von den stetigen Grössen bezeichnet werden, welcher die

Grössen nicht als unabhängig von der Lage existirend und durch einander messbar betrachtet, sondern von den Massverhältnissen ganz absehend, nur ihre Orts -- und Gebietsverhältnisse der Untersuchung unterwirft.[23]

(In the investigation of functions which arise from the integration of total differentials several theorems belonging to analysis situs are almost indispensible. With this name, used by Leibniz, perhaps not with quite the same meaning, one may denote that part of the theory of continuous magnitudes which considers such magnitudes not as existing independently of their position or as measurable by one another, but which investigates only their local and regional properties entirely divorced from measure-relations.)

1.5 SOME FUNDAMENTAL DIFFICULTIES WITH RIEMANN'S APPROACH TO THE FORMULATION OF A NON-METRICAL ANALYSIS SITUS

We saw that after the beginning of the nineteenth century mathematicians could no longer attribute to classical geometrical intuition unlimited confidence. In his Ausdehnungslehre Grassmann presented a calculus of extension in which the notions of number and geometrical objects were not even presupposed but were derived from continuous quantities. Thus, ten years later Riemann did not speak of "points" but of "determinations" (Bestimmungsweisen) in his description of the n-dimensional manifold and argued that in a continuous n-dimensional manifold the "metrical relations" (Massverhältnisse).

> . . . lassen sich nur in abstracten Grössenbegriffen untersuchen und im Zusammenhange nur durch Formeln darstellen; unter gewissen Voraussetzungen kann man sie indess in Verhältnisse zerlegen, welche einzeln genommen einer geometrischen Darstellung fähig sind, und hierdurch wird es möglich, die Resultate der Rechnung geometrisch auszudrücken. Es wird dabei, um festen Boden zu gewinnen, zwar eine abstracte Untersuchung in Formeln nicht zu vermeiden sein, die Resultate derselben aber werden sich im geometrischen Gewande darstellen lassen.[24]

> (. . . can be investigated only in terms of abstract concepts of magnitude, and their independence is exhibited only through formulas. Under certain assumptions, however, one can resolve them into relations which are individually capable of geometric representation, and in this way it becomes possible to express the results of calculation geometrically. Thus, although an abstract investigation with formulas certainly cannot be avoided, the result can be presented within a geometrical framework.)

It is clear that Riemann attempted to develop a manifold concept of great generality and that he thought that his description and analysis of n-dimensional manifolds relied on intuitive considerations only to the extent of justifying the introduction of local coordinates.[25] Apart from this, Riemann seems to have felt that he stood on the solid ground of Analysis. Analysis is however ultimately based on the concept of real number, which, up to Riemann's time, had remained on a very intuitive basis.[26] It would seem then that the concepts of continuity and differentiability that were implicitly involved in Riemann's definition of dimensionality, by specifying the number of coordinates needed to determine a "point", could only have had an intuitive informal meaning for him.

Riemann's whole enterprise seemed to suffer from a serious difficulty. Riemann's informal procedure of coordinatizing the n-dimensional manifold seemed to reduce the latter to the n-dimensional number space. His attempt to construct a non-metrical analysis situs was thus seen to be logically circular.[27] It appeared that the coordinates of the points of a manifold are endowed right from the start with the usual Euclidean distance measure, thereby imposing a metric structure on the topological manifold. How would it thus be possible for Riemann to argue that a continuous n-dimensional manifold admits different possible metrical relations?

Toward the end of the nineteenth century an analogous problem arose in connection with projective geometry. If projective geometry is fundamental to Euclidean geometry, is it permissible to use a coordinate system which is usually defined by a metric? Would not the definition of cross ratios and therefore concepts based on cross ratios depend on the notion of length? But the concept of length plays no role in projective geometry. Length is not an invariant under arbitrary projective transformations. Felix Klein solved this problem by means of von Staudt's quadrilateral construction and showed how to coordinatize space without involving metric assumptions.[28] Klein was also acutely aware of the logical circularity which threatened Riemann's enterprise and says:

> Es folgt, dass alle Untersuchungen, welche mit den Begriffen Zahlenmannigfaltigkeit und differentiierbare Funktionen beginnen, wenn man sie direkt als Untersuchungen über die Grundlagen der Geometrie interpretieren wollte, einen Zirkel enthalten würden.[29]

> (It follows that if all investigations which begin with the concepts of number manifold and differentiable functions were to be interpreted as foundations of geometry, they would contain a circle.)

Although Riemann did not provide an explicit formal procedure for assigning coordinates to the manifold, he certainly must have thought that coordinate assignments were arbitrary. Perhaps he implicitly assumed something like a generalization of Gauss's arbitrary parametric coordinatization for two-dimensional surfaces. What is required nonetheless, and what is absent in Riemann's approach that would show the complete arbitrariness of the coordinate assignments, and thereby establish the possibility of a non-metrical analysis situs on a non-circular basis, is the notion of homeomorphic equivalence between the open subsets of the manifold and of the number space. Moreover, for a manifold to be a differentiable manifold the homeomorphisms must in addition satisfy certain differential compatibility conditions. It is clear that Riemann intends his manifolds to be differentiable manifolds, since he assumes differentiability in the later sections of his lecture.

Another difficulty concerns the problem of topological invariance of dimensionality: A sufficiently small neighborhood of some point p in an n-dimensional manifold may be mapped one-to-one and continuously upon an open neighborhood of the n-dimensional number space $\mathbb{R}^n$. The points in the latter consist of n-tuples of real numbers $\langle x^1(p), x^2(p), \ldots, x^n(p)\rangle$. Then any one-to-one transformation of the coordinates

$$\bar{x}^i(p) = \phi^i(x^1(p), x^2(p), \ldots, x^n(p); \quad (i=1,\ldots,m)$$
$$x^j(p) = \psi^j(\bar{x}^1(p), \bar{x}^2(p), \ldots, \bar{x}^m(p); \quad (j=1,\ldots,n)$$

constitutes a new coordinate assignment representing the same neighborhood. The question is whether or not n is necessarily equal to m.

Shortly after the publication of Riemann's lecture, Cantor discovered the famous theorem which says that an n-dimensional cube can be put into one-to-one correspondence with a line segment.[30] In other words $\mathbb{R}$ and $\mathbb{R}^n$ are equipotent. This seemed to undermine Riemann's informal concept of dimension based on the number of coordinates required to determine a point. Cantor's counterintuitive result further strengthened the motivation to put geometry and topology on a solid foundation and to free them entirely from all dependence on intuition.

In a remark to Cantor, Dedekind[31] suggested that if $m \neq n$, then it should be possible to prove the impossibility of a bicontinuous one-to-one correspondence between $\mathbb{R}^n$ and $\mathbb{R}^m$. Thus Dedekind understood the underlying reason of Cantor's counterintuitive result and realized that proving dimensional invariance requires the proof of some theorem that explicitly exploits the concept of continuity.

The path to the solution of this problem turned out to be a long and treacherous one. It was not until 1911/12 that Brouwer and Poincaré provided a satisfactory solution.[32]

FOOTNOTES

1. [12], pp. 272-287.
2. [12], p. 547.
3. [12], p. 112, (Nachträge).
4. [12], p. 547.
5. All translations from German texts cited are my own.
6. [12], p. 548.
7. [12], p. 549.
8. [12], p. 272.
9. [12], pp. 272-273.
10. [12], p. 286.
11. [7], p. 124.
12. [12], pp. 3-48.
13. [12], p. 274.
14. [7], p. 123.
15. [12], p. 274.
16. [5].
17. [9].
18. [6].
19. [1], p. 313.
20. [10], p. 50.
21. [12], pp. 273-274.
22. [7], p. 125.
23. [12], p. 91.
24. [12], p. 276.
25. The expression "intuitive consideration" must be under-

stood here as applying strictly speaking only for $n \leq 3$. What is involved for $n > 3$ is really an argument by formal analogy. (See [1], p. 311, (footnote)).

26. [1], p. 311.
27. See Klein [8], pp. 388-389 and Russell [13], pp. 118-119; pp. 123-126.
28. [8], pp. 375-379.
29. [8], p. 389.
30. [4], p. 25.
31. [4], p. 38.
32. Brouwer [2], pp. 161-165 and [3], pp. 55-56. Poincaré [11], pp. 486-487.

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CHAPTER 2

HYPOTHESES OR FACTS: RIEMANN AND HELMHOLTZ'S SEARCH FOR A METRIC

2.1 RIEMANN'S ANALYTIC JUSTIFICATION

Riemann's Conceptual Separation Thesis, as cited in the beginning of Section 1.4, states that the concept of a continuous manifold does not by itself involve the specification or existence of a metrical structure. That is, a specific metric cannot be defined on the basis of the topology of a continuous manifold.

It should be noted, that in that passage Riemann takes it as *given* whether a manifold is *continuous* or *discrete* and addresses himself to the question of how metrical relations are to be determined for either case. That is, Riemann does not inquire into an underlying reason or basis of the continuity or discreteness properties of a manifold, but takes these topological characteristics as essential and given.

Any manifold M which is metric admits some distance function. Let f: $M \times M \rightarrow \mathbb{R}$ be a function on ordered pairs of members of a non-empty set M into the reals. Then f is a distance function if and only if it fulfills the following axioms: For all p, q, r ε M

i) $f(p,q) \geq 0$ and $f(p,p) = 0$

ii) $f(p,q) = f(q,p) > 0$, for $p \neq q$

iii) $f(p,q) + f(q,r) \geq f(p,r)$

A manifold on which a distance function is defined is a metric manifold. It should be noted that nothing so far has been said about the special form of this function. The axioms constitute a set of conditions or constraints which only partly determine the extension of the concept 'distance'. They are the minimal conditions which <u>any</u> distance function and hence any metric has to satisfy. Hence there can be different metrics corresponding to different distance functions which satisfy the constraints but assign different distances to one and the same interval.

Since continuous manifolds were of prime interest to him, Riemann examines in Part II of his lecture, how metrical relations can be determined on a continuous manifold, what specific form $f: M \times M \rightarrow \mathbb{R}$ should have.

Part II is entitled: "Metric relations of which a manifold of n-dimensions is susceptible, on the assumption that lines have a length independent of their configuration, so that every line can be measured by every other." It is useful to contrast Riemann's method of assigning length with the method that one normally uses in analytic geometry when one assigns lengths to curves in the three dimensional manifold. One starts with the notion of distance between pairs of points, that is, one first assigns a length to straight lines. The length of other lines is then given as the least upper bound of inscribed curves that are made up of straight line elements. In the limit this process reduces to integration.

Riemann, on the other hand, develops a uniform method of assigning lengths to all curves in a manifold. This method, which does not depend on first distinguishing a particular class of curves (straight lines), measures the length of tangent vectors, and the length of curves can then be defined by an integral.

Let p be a point of the manifold M that is characterized by an n-tuple of coordinates $x^i(p)$. If $x^i(p) + dx^i(p)$ are the coordinates of a neighboring point p' then the length of the distance ds between p and p' must be some function f_p at p of the differential increments $dx^i(p)$, viz., $ds = f(dx^1(p), \ldots, dx^n(p))$. Riemann suggests that such a function should satisfy the following requirements:

(a) Functional Homogeneity Requirements: If the increments $dx^i(p)$ increase in the same proportion, then ds must increase accordingly.

In other words, f_p should at least be a homogeneous function of first degree in the differentials so that multiplication of each $dx^i(p)$ with the same real proportionality factor λ corresponds to the multiplication of f_p with λ. That is, for $\lambda > 0$, and $ds = f(x^i(p), dx^i(p))$, the following should hold:

$$f(x^i(p), \lambda dx^i(p)) = \lambda f(x^i(p), dx^i(p)) = \lambda ds$$

(b) Requirement of sign invariance: A change in sign should leave the value of ds invariant.

Clearly this condition is satisfied by every positive homogeneous function of degree 2n (n = 1, 2, 3, . . .). The simplest case occurs when $n = 1$. The length element ds is then the square root of the homogeneous function of second degree and can be expressed in standard form as

$$ds = \sqrt{dx^1(p)^2 + \ldots + dx^n(p)^2} \; .$$

That is, at each point of M there exists a unique coordinate system in which the square root of the homogeneous function of second degree can be expressed in the above standard form. Since a measure of length should be invariant with respect to all coordinate transformations, Riemann's well known general expression for the measure of length at p is given by

$$ds^2 = \sum_{ij}^{n} g_{ij}(p)\,dx^i(p)\,dx^j(p)$$

where the components of the metric tensor satisfy the symmetry condition $g_{ij} = g_{ji}$.

For a curved 2-dimensional surface we evidently have

$$ds^2 = g_{11}(p)(dx^1(p))^2 + 2g_{12}(p)dx^1(p)dx^2(p)+g_{22}(p)(dx^2(p))^2$$

which corresponds to the Gaussian formula for the line elements in a 2-dimensional hypersurface, viz.,

$$ds^2 = E(dx^1)^2 + 2Fdx^1dx^2 + G(dx^2)^2$$

where the E, F, G now represent the component of the metric tensor of the surface. That is,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \; (g_{ij} = g_{ji}) ,$$

and the invariance of length under coordinate transformation is satisfied because of the tensorial character of the g_{ij}.

The assumption that $ds^2 = f_p^2$ is a quadratic differential form is not only the simplest one, but is also the preferred one for other important reasons which we shall now discuss. It should be noted, however, that Riemann points out that it is also to save time and to allow geometric descriptions of his results that he restricts his attention to the special and simple case of n = 1. Riemann was well aware of the possibility that one could also consider ds as the 4th root of a homogeneous polynomial of 4th order in the differentials. Riemann says:

> Der nächst einfache Falle würde wohl die Mannigfaltigkeiten umfassen, in welchen sich das Linienelement durch die vierte Wurzel aus einem Differentialausdrucke vierten Grades ausdrücken lasst. Die Untersuchung dieser allgemeinern Gattung würde zwar keine wesentlich andere Principien erfordern, aber ziemlich zeitraubend sein und verhältnismässig auf die Lehre vom Raume wenig neues Licht werfen, zumal da sich die Resultate nicht geometrisch ausdrücken lassen; ich beschränke mich daher auf die Mannigfaltigkeiten, wo das Linienelement durch die Quadratwurzel aus einem Differentialausdruck zweiten Grades ausgedrückt wird.[1]

> (The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essentially different principles, but it would be rather time consuming and throw relatively little new light on the study of space, especially since the results cannot be expressed geometrically; I consequently restrict myself to those manifolds where the line element can be expressed by the square root of a differential expression of the second degree.)

Analytically, Riemann motivated the restriction to the Pythagorean case in the following way. Let S_p be a circle around some point $p \varepsilon M$ which consists of the geometric positions of all points p' which have a determinate distance D from p measured along the shortest lines. That is $S_p = \{p' | \min|p-p'| = D\}$. Then the set $\{S_p^\alpha\}_{\alpha \varepsilon I}$ of such circles around $p \varepsilon M$ may be expressed analytically as $F_p(x^1, \ldots, x^n) =$ constant. A Taylor expansion on F_p around p begins then with quadratic terms. Riemann disregards the higher terms and considers only the quadratic ones, which together constitute a quadratic form that is always ≥ 0. If this form does not vanish and is > 0 in all directions then one obtains the Pythagorean form of ds.

Weyl[2] points out, however, that one should not put too much weight on Riemann's justification for the pythagorean form of the distance function f_p. Riemann's arguments for disregarding the higher terms in the Taylor expansion are artificial and not compelling. Weyl suggests another justification which is essentially based on the notion of 'measurability'. According to Weyl, it is reasonable to assume that the nature of the metric is everywhere the same. That is, it is reasonable to require that if at some arbitrary point $p \varepsilon M$, ds is given through an expression $f_p(dx^1, \ldots, dx^n)$, where f_p is a homogeneous (but not necessarily rational) function of first order in its arguments, then the functions f_p at any other point $p \varepsilon M$ all belong to a unique equivalence class [f], in the sense that they all arise from such a function f (where f is the representative member of the equivalence class) through homogeneous transformations of its arguments. This is not in general the case when f_p is the fourth root of a differential form of fourth degree alluded to earlier. In the case when f_p is pythagorean in form this requirement is fulfilled, since every positive-definite quadratic form can be obtained through a homogeneous linear transformation of the standard expression $f_p = \sqrt{(dx^1)^2 + \ldots + (dx^n)^2}$. To every possible equivalence class [f] of homogeneous functions corresponds a <u>type</u> of metrical space. Among these types of possible metrical spaces the Pythagorean-Riemannian space, for which $f_p^2 = (dx^1)^2 + \ldots + (dx^n)^2$, is one among several types of possible metrical spaces. The problem, therefore, is to single out the equivalence class [f], where f corresponds to $f_p^2 = (dx^1)^2 + \ldots + (dn^n)^2$, from other possibilities and to provide arguments for this preference.

2.2 HELMHOLTZ'S JUSTIFICATION FROM EXPERIENCE

The first satisfactory justification of the Pythagorean-Riemannian metric form, though, as we shall see, limited in scope, was provided by the investigations of Hermann von Helmholtz[3]. Helmholtz's justification was later developed and improved group-theoretically by Sophius Lie.[4] Helmholtz had been working on the physico-psychological problem of space for some time and was very surprised, when reading some account of Riemann's work in two obituaries by Ernst Schering, that his own work in space perception was closely related to the geometrical concerns of Riemann. He quickly obtained a copy of the Habilitationsvortrag from Schering and published two papers on the foundation of geometry. The first paper entitled "Über die Thatsächlichen Grundlagen der Geometrie" (1866) is a general discussion and less mathematical in character. It was expanded both in thematic scope and mathematical rigour in a second paper, "Über die Thatsachen, die der Geometrie zum Grunde liegen" (1868).[5]

Helmholtz's justification of the Pythagorean form of the metric does not even require the measurability of line elements. In defining the measure relation on the manifold, Helmholtz diverges from Riemann's analytic approach and makes use merely of the fundamental concept of geometry, namely, the concept of a <u>congruent mapping</u>, and characterizes the structure of space by requiring of space the full homogeneity of Euclidean space. That is, Helmholtz's point of departure is his claim that all our knowledge of space comes from the observation of the properties of rigid bodies, and that consequently the general properties of space should be deducible from the conditions that are satisfied by rigid bodies in motion. Abstracting from our experience of the movement of rigid bodies, Helmholtz was able to mathematically derive Riemann's distance formula.

Essentially, Helmholtz asked the question: What must the geometric structure of space be, in order that a mechanics of rigid bodies is realizable in that space, and what must the geometry of space be like in order that it admits the conservation law of momentum and angular momentum as integrals of the equation of motion.

Helmholtz discovered that the free mobility, that is, arbitrary change of location in the form of translation and rotation without dilatation (displacement and rotation) of rigid bodies, is the necessary and sufficient condition for the formalizability of the conservation law of momentum and angular momentum, and that space possesses these properties exactly then when it possesses a constant (imaginary, real or vanishing) curvature.

The requirement, that the geometry of space be such that it admits arbitrary change of location in the form of translation and rotation without dilatation, and hence admits rigid body motion, led Helmholtz to the characterization of space with a Riemannian metric structure and constant curvature.

Riemann and Helmholtz shared a common concern in their investigation of the nature of geometry. Both were concerned to clarify the distinction between assumptions that are logical consequences of any manifold, and those assumptions which restrict the range of structural possibilities implied by the manifold to the special case of physical three-dimensional space. Both agreed on the fundamental importance of the measure relations in determining the geometrical structure of physical space. The basic difference in their approach to this determination displays, however, an important difference in their philosophical attitude toward empirical geometry.

In contrast to Riemann's analytic orientation to the measure determination of space, Helmholtz was mainly concerned with the space of everyday experience. He started out with the assumption of rigid bodies and their free mobility, because he saw a direct connection between <u>movement</u> and the idea of <u>congruence</u>; he considered the notion of congruence to be more basic in our perception of space than the equation of the shortest distance between two points. Helmholtz suggested that a blind man can understand geometry without the latter but not without the former.

Of course Helmholtz did not dispute Riemann's derivation of the distance formula, which, from an analytical point of view, represents the simplest choice. But for Helmholtz it is experience and not analytic simplicity which determines the geometric structure of physical space. His results were motivated by his concern with physiology, that is, with facts which are fundamental to our perceptual experience of space.

For Helmholtz then, the structure of space follows from the possibility of congruent transport of rigid bodies; that is, the structure of space constitutes a necessary condition for the possibility of the realizability of certain physical processes and operations within that space. Whether or not space possesses a constant curvature, or whether or not space is a general Riemannian space with variable curvature, depends on whether or not physics allows the introduction of ideal rigid bodies.

Helmholtz thought that our notion of 'length' is based on congruence, and that congruence in turn is based on rigid body motion. He considered the homogeneity postulate underlying the free mobility of rigid body motion to be a necessary pre-condition for the very possibility of geometrical knowledge, without which physical geometry, and hence our knowledge of it, would presumably break down.

> However, in general one cannot speak of congruence, if rigid bodies or point systems cannot be moved to another place without change of form, and if congruence of two spatial magnitudes is not a datum which remains independent of all movements. The possibility of space measurement through the establishment of congruence, I thus presupposed from the beginning and set myself the problem to look for the most general analytic form of a multiply determined manifoldness in which the required form of movement is possible.[6]

2.3 THE RELATION BETWEEN HELMHOLTZ AND RIEMANN

Helmholtz's substitution of "facts" for "hypotheses" in his title indicates how he understood his contribution in its relation to Riemann's work. It carries the implication that Helmholtz limits his foundational theory to the possibilities implied by the fundamental presuppositions of experience, thus disallowing other possibilities which Riemann left open. Riemann thought that the homogeneity postulate for rigid body motion, and the geometry entailed by it, may not hold strictly but only approximately. Helmholtz's fundamental presuppositions of experience are according to Riemann hypotheses: ". . . one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small".[7]

Indeed from the standpoint of modern physics one tends to reverse the logical sequence. The existence of rigid bodies and hence rigid body motion is not a fundamental self-evident requirement anymore, but only a special property that obtains in a definite limiting case of physics, namely, classical mechanics and geometry. Helmholtz's reasoning only shows that the possibility of this limiting case entails the Riemannian metric.

With remarkable forsight, Riemann did consider the possibility that the "systems of facts" which underlie

Euclidean and (ordinary) non-Euclidean geometry, namely, the group of rigid body motion, may break down, and approached the problem of physical geometry from a much more general basis, which includes classical mechanics and "distance geometry" as a limiting case.

The difference between Euclid's elements and ordinary non-Euclidean geometry consists only in the discarding of the axiom of parallelism. Helmholtz considered only Euclidean and ordinary non-Euclidean geometries by assuming free mobility of ideal rigid bodies.

Riemann, on the other hand, assumed the validity of the Pythagorean metric only in the infinitely small. His geometric assumptions rest on infinitesimally small magnitudes. It is essentially a geometry of infinitely near points and conforms to the Leibnizian idea of the Continuity Principle: All laws are to be formulated as field laws and not by action-at-a-distance.

In contrast to the action-at-a-distance laws, the close-action-laws (that is, strictly speaking "field-laws") relate the field magnitudes only to infinitesimally neighboring points in space. The principle of the existence of field effects requires that the value of some field magnitude Ψ_1 at each point (x, y, z) depends only on the values of the other field magnitudes $\Psi_2, \Psi_3, \ldots, \Psi_n$ at the same point of space, i.e.,

$$\Psi_1(x,y,z) = f(\Psi_2(x,y,z), \ldots, \Psi_n(x,y,z))$$

or

$F(\Psi_1, \Psi_2, \ldots, \Psi_n) = 0$

The field quantities consist of partial derivatives of position functions. But this requires the knowledge of the behavior of the latter only with respect to the neighborhood of some point in space. It is important to understand that the neighborhood may be infinitesimally small. Hence in order to construct the field law only the behavior of the world in the infinitesimally small is required. Thus, it is more accurate to speak of "field-laws" instead of "close-action-laws", in order to avoid thinking of the latter as laws which hold for small distances only.

The electric action in Faraday's proposed interaction model is not regarded as somehow reaching across the spatial separation of two charged particles. Rather, one understands the interaction as an <u>infinitesimal</u> interaction between the charged body and its surrounding field. A charged body feels a force because it is in <u>contact</u> with its surrounding field. That is, each charged body feels the field of the other as a <u>local</u> force.

The connection with Riemann's infinitesimal geometric standpoint is this. Just as Faraday's field interpretation of electric phenomena may be contrasted with action-at-a-distance interpretations, so Riemann's <u>infinitesimal geometric standpoint</u> may be contrasted with "distance-geometry", namely, Euclidean and ordinary non-Euclidean geometry. Riemann adopted the same principles in geometry as did Faraday and Maxwell before him within physics, specifically electromagnetism, namely, to understand the world from its behavior in the infinitely small. Thus, Riemann's radical departure from all previous geometric theories consists in introducing the standpoint of a "near-geometry" which is analogous in spirit and method to "near-action-physics" introduced by Faraday and developed by Maxwell. In his "Die Grundlagen der Physik" David Hilbert says:

> Euclidean geometry is a distance-law that is unfamiliar to modern physics: While the theory of Relativity rejects Euclidean geometry as a general presupposition for physics, it teaches that geometry and physics are of similar character and as one science rests on the same foundation.[8]

Later, we shall see that when Weyl speaks of Struckturfelder (structural fields) over a manifold, or when he says that a manifold carries a certain geometrical field, he intends to characterize and understand Riemann's "near-geometry", as a geometry of geometric fields over the manifold.

It was essentially for reasons of simplicity and time that Riemann restricted himself to express the infinitesimal distance ds through a homogeneous function of second degree in the coordinate differentials. Riemann then characterized this metric manifold through its curvature properties, which in the limiting case reduce to the Gaussian curvature constant K. It is after these purely mathematical considerations that we come to Riemann's arguments (already alluded to earlier) which essentially characterize his position concerning the problem of empirical geometry: The geometry of continuous physical space cannot be established purely on the basis of the conceptual level of a general metrical manifold. Rather, the restrictive conditions, which single out the specific metrical structure of physical space from other possible distinct metrical structures compatible with the manifold topology, must be taken from experience. It is at this juncture that those "hypotheses" in the title of his address emerge. Already in the Introduction Riemann points out that the systems of facts (Thatsachen) of Euclidean as well as those of ordinary non-Euclidean geometries are

> . . . nicht nothwendig, sondern nur von empirischer Gewissheit, sie sind Hypothesen; man kann also ihre Wahrscheinlichkeit, welche innerhalb der Grenzen der Beobachtung allerdings sehr gross ist, untersuchen und hiernach über die Zulässigkeit ihrer Ausdehnung jenseitz der Grenzen der Beobachtung, sowohl nach der Seite des Unmessbargrossen, als nach der Seite des Unmessbarkleinen urtheilen.[9]
>
> (. . . not necessary, but only of empirical certitude, they are hypotheses; one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small.)

And in Part III, "Application to Space", Riemann says:

> Setzt man voraus, dass die Körper unabhängig vom Ort existiren, so ist das Krümmungsmass überall constant, und es folgt dann aus den astronomischen Messungen, dass es nicht von Null verschieden sein kann; . . . Wenn aber eine solche Unabhängigkeit der Körper vom Ort

nicht stattfindet, so kann man aus den Massverhältnissen im Grossen nicht auf die im Unendlichkleinen ~~schliessen~~; es kann dann in jedem Punkte das Krümmungsmass in drei Richtungen einen beliebigen Werth haben, wenn nur die ganze Krümmung jedes messbaren Raumtheils nicht merklich von Null verschieden ist . . . Nun scheinen aber die empirischen Begriffe, in welchen die räumlichen Massbestimmungen gegründet sind, der Begriff des festen Körpers und des Lichtstrahls, im Unendlichkleinen ihre Gültigkeit zu verlieren; es ist also sehr wohl denkbar, dass die Massverhältnisse des Raumes im Unendlichkleinen den vorausssetzungen der Geometrie nicht gemäss sind, und dies würde man in der That annehmen müssen, sobald sich dadurch die Erscheinungen auf einfachere Weise erklären liessen.[10]

(If one assumes that bodies exist independently of position [this is of course Helmholtz's view, H.K.], then the measure of curvature is everywhere constant, and it then follows from astronomical measurements that that measure cannot be different from zero . . . But if such an independence of bodies from position does not exist, then one cannot draw conclusions about metric relations in the infinitely small from those in the large; at every point the curvature can then have arbitrary values in three directions if only the total curvature of every measurable portion of Space is not perceptibly different from zero . . . Now it seems that the empirical concepts on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore very well conceivable that the metric relations of Space in the infinitely small are not in accord with the presuppositions of [ordinary, H.K.] geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.)

It should be noted that the beginning sentences of the above citation indicate that Riemann was fully aware of the fact that free mobility of rigid bodies is possible in a Riemannian manifold if and only if its curvature is uniform. Therefore, Helmholtz's significant contribution does not consist in the demonstration that a Riemannian metric space of constant curvature is a sufficient condition for the homogeneity postulate; rather, it consists in the demonstration that a Riemannian metric is a necessary condition for it.

For Helmholtz our ordinary middle sized geometrical knowledge rests on the "Thatsache" of the free mobility of solid bodies. This points to his different methodological departure in contrast to Riemann who speaks of "Hypothesen". Riemann's prophetic remarks that the concept of solid bodies may break down in the small suggest that for him geometric structure in the small involves dynamical considerations; that is, it involves taking into consideration the atomic constitution of matter and hence conceiving of solidity as an equilibrium of atomic configuration.[11]

> Auf der Genauigkeit, mit welcher wir die Erscheinungen in's Unendlichkleine verfolgen, beruht wesentlich die Erkenntniss ihres Causalzusammenhangs . . . Die Fragen über die Massverhältnisse des Raumes im Unmessbarkleinen gehören also nicht zu den müssigen.[12]
>
> (Our knowledge of the causal connections of phenomena depends essentially on the exactness with which we pursue phenomena into the infinitely small . . . Questions about the metric relations of Space in the immeasurably small are thus not idle ones.)

Of course Riemann's conjecture concerning the structure of the world "in the small" has not been born out by the Theory of General Relativity. There is no indication of strong variability of spacetime curvature for spacetime regions of microscopic scale that would average out with respect to ordinary bodies such that their solidity would be an equilibrium of atomic configuration. It is rather in the "large" that the metric relations are not in agreement with the presuppositions of ordinary geometry.

Riemann and Helmholtz are both Empiricists with regard to our knowledge of empirical geometry. The essential difference between them, however, is this: While Helmholtz takes the homogeneity postulate of rigid body motion as a fundamental presupposition in our perception of space and our knowledge of physical geometry, Riemann seeks to explore the widest range of possibilities and therefore starts from a much more general basis. Allowing for the possibility of an inhomogeneous physical metrical structure that is causally dependent on an inhomogeneous distribution of matter, Riemann introduced the infinitesimal standpoint into geometry. Analogous to the "close-action" physics of Faraday and Maxwell, Riemann's "near-geometry" is an attempt to understand the world from its behavior in the small; it introduces into the study of geometry the notion of <u>structural</u> or <u>geometrical fields</u>.

Riemann recognized that the possible empirical sources of information that provide epistemic access to the geometrical structure of space may be intimately connected to our physical theories. We must therefore not take too narrow a view as to which system of data will prove relevant in providing information of the geometrical structure which space actually has. We cannot hope to foresee what empirical data will prove relevant. As a mathematician Riemann sets himself the task to explore the widest range of possibilities which includes, among others, the case in which geometric structure, particularly in the small, is intimately bound up with our whole theoretical understanding of physical interaction.[13] At the very end of Part III (Application to Space) he says:

> Die Entscheidung dieser Fragen kann nur gefunden werden, indem man von der bisherigen durch die Erfahrung bewährten Auffassung der Erscheinungen, wozu Newton den Grund gelegt, ausgeht und diese durch Thatsachen, die sich aus ihr nicht erklären lassen, getrieben allmählich umarbeitet; solche Untersuchungen, welche, wie die hier geführte, von allgemeinen Begriffen ausgehen, können nur dazu dienen, dass diese Arbeit nicht durch die Beschränktheit der Begriffe gehindert und der Fortschritt im Erkennen des Zusammenhangs der Dinge nicht durch überlieferte Vorurtheile gehemmt wird.[14]
>
> (An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been proved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices.)

Immediately preceding the above passage occurs Riemann's famous statement concerning the possibility of a dynamical geometry:

> Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Massverhältnisse des Raumes. Bei dieser Frage, welche wohl noch zur Lehre vom Raume gerechnet werden darf, kommt die obige Bemerkung zur Anwendung, dass bei einer discreten Mannigfaltigkeit das Princip der Massverhältnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muss. Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenden Kräften gesucht werden.[15]

> (The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying space must form a discrete manifold, or the basis for the metric relations must be thought outside it, in binding forces acting upon it.)

The above statement expresses what I shall call Riemann's Dynamical Hypothesis (RDH). It was brought to a concrete realization fifty years later in Einstein's Theory of General Relativity. Its philosophical significance within the context of the realism/conventionalism debate will be one of the main concerns in Part II.

FOOTNOTES

1. [6], p. 278.
2. [8], p. 29.
3. See [1], [2] and [3].
4. [5], especially pp. 393-543.
5. [2].
6. [2], p. 621.
7. [6], p. 273.
8. [4], p. 278.
9. [6], p. 273.
10. [6], p. 285.
11. [7], p. 25.
12. [6], p. 285.
13. [7], p. 25.
14. [6], p. 286.
15. [6], pp. 285-286.

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CHAPTER 3

RIEMANN AND GEOMETRICAL CONVENTIONALISM

3.1 INTRODUCTION

What is the philosophical significance of Riemann's bold and, as it later turned out, fruitful conjecture, that "either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside, in binding forces acting upon it"? In particular, how is one to understand the nature of this alleged causal/dynamical relation between metrical forces on the one hand and physical geometry or metrical relations on the other?

Descartes' strict distinction between the res cogitanse and the res extensa meant the strict separation of the world of the mental, the world of mathematics as well as geometry, from the world of physical reality. It is difficult to see how material forces can have an effect on that which, according to this classical dualist view, has its ontological basis in the "mental realm."

From a monistic substance point of view one might be tempted to see in Riemann's remark the suggestion that mental structures involving geometrical relations and physical reality do not, as they did for Descartes, constitute the attributes of two independent basic substances, but represent in Spinozistic terms two attributes of one underlying substance. According to this viewpoint then, physical forces do not exert effects on the geometer's consciousness, rather the relations between geometry and physical forces are grounded in one underlying substance.

Of course, the latter interpretation rests on the traditional substance metaphysics and can hardly be considered an adequate clarification of Riemann's remarks. Such an interpretation fails to clarify and make understandable precisely that which is of interest, namely, the nature of the relation between geometry and physical forces or physical reality. Instead of explaining this relation one simply asserts that no fundamental difference exists by saying that geometry and physical reality constitute two different aspects of one underlying reality.

It would seem then that Riemann's conjecture of a reciprocal/dynamical interaction between material forces and metrical relations can only be understood as asserting a possibility which requires for its physical realization that there exist, in some as yet unspecified sense of existence, physically real geometrical and hence physically real metrical relations.

Before offering a more substantial account of the significance of Riemann's Dynamical Hypothesis within the context of the realism/conventionalism debate, I shall first provide a general statement of geometrical conventionalism apart from issues of its alleged historical ancestry as seen by Grünbaum. This will be followed by a critical examination of Grünbaum's unyielding insistence that Riemann's work must be seen as the immediate progenitor of his brand of geometric conventionalism. I shall show that Grünbaum's exegesis of Riemann's inaugural lecture does not hold. A careful examination of Riemann's text shows that it does not support Grünbaum's interpretation. In fact, it will turn out that Grünbaum's interpretation is plainly at odds with the basic intentions of Riemann. This is particularly apparent when one takes into account the historical context of the Habilitationsvortrag as discussed in Part I.

3.2 GEOMETRIC CONVENTIONALISM

There is a long-standing philosophical tradition referred to as geometrical conventionalism which goes back at least to Poincaré. Its central claim is the Indeterminacy Thesis which says that there is no such thing as <u>the</u> geometry.

Broadly speaking, the doctrine takes spacetime as being factually definite in its local differential topology but considers all additional geometrical structures, such as the projective, conformal, affine and metric structures, as being underdetermined by all possible evidence, by all the actual or possible trajectories and coincidences of physical objects within spacetime. The reason for this underdetermination is seen in the fact that while the physical geometry postulated by a physical theory supposedly pertains to the properties of the spacetime manifold, any description of such a geometry must always be supplied in terms of the behavior of material systems within spacetime, different assumptions about their behavior leading to different geometric descriptions. In particular, the metrical structure ascribed to physical spacetime depends on how the mathematical concept of 'congruence' is interpreted by being correlated with the coincidence behavior of physical congruence standards under transport. According to geometrical conventionalism there is no fact of the matter with which to arbitrate between different interpretations of 'congruent'.

Geometric conventionalism invites us to recognize that geometric predicates which give the appearance of being monadic predicates signifying properties of the point-structure of the spacetime manifold, really signify relations. The congruence relations of two regions of spacetime do not constitute true relations that just pertain to two spacetime regions, but constitute relations between these regions and physical metrical standards. Just as things do not have true names, since a name is not a monadic property of a thing so named, but is a relation between the thing named and the speaker, so physical geometrical properties are relational in character, and different seemingly incompatible geometrical propositions are equally true.

The above remark must not be construed, however, as suggesting that geometrical propositions of physical spacetime are intended to be conventional in the sense in which every word of a language is conventional in that its extension can always be redefined. What is not intended is that kind of conventionality which concerns those decisions that associate semantic content with a particular sound or sign. This would be what Grünbaum calls trivial semantic conventionalism according to which the truth values of geometrical propositions depend on semantic conventions. Any proposition depends for its truth value not merely upon extra-linguistic facts, but for any given such fact, a proposition will assert or deny that fact only if its terms have certain meanings. We can turn a false into a true factual claim by conventionally reassigning the meanings of the relevant terms. The truth of any proposition is, in this sense, dependent on the semantic conventions we adopt.

Model-theoretically speaking, a physical geometry obtains by giving a physical interpretation to all the relevant terms of an abstract uninterpreted calculus. But this does not capture the essence of the conventionality thesis of physical geometry. The thesis of the conventionality of 'congruence' is an assertion which concerns the geometrical properties of spacetime. It is not about the application of the sign or sound 'congruent', but about the non-semantic question whether physical space or spacetime does or does not possess an intrinsic metric prior to some specified coordinating definition of 'congruence'.

The sense of conventionality intended here arises because of the alleged factual absence of certain objective features in the world that would be sufficient to uniquely determine the extension of the relevant theoretical terms. Non-trivial conventionalism arises when one is faced with a decision to choose from several alternatives and when, in the factual absence of relevant objective features of the world, there is no factual objective basis on which to make such a choice.

It is useful to distinguish in this context between epistemological and ontological conventionality. While the world may or may not lack definite objective features that would be sufficient to uniquely determine the extension of theoretical terms and thus provide an empirical basis for choosing from amongst several possible alternatives, we either do not know or cannot know of those features. Thus epistemological conventionality leaves open whether the extension of some theoretical term is or is not in fact underdetermined. Epistemological conventionality does not therefore preclude a realistic interpretation of the theoretical terms in question. But we either do not know or cannot know whether or not the world has the relevant objective features that would support such an interpretation.

Ontological conventionalism makes the factual claim that the world lacks certain objective features that would be sufficient to uniquely determine the extension of certain relevant theoretical terms. Ontological conventionalism certainly entails epistemological conventionalism. The absence of certain relevant objective features of the world trivially entails our corresponding lack of knowledge of them. But the converse need not be true. Lack of knowledge of certain relevant objective properties of the world does not entail their absence.

3.3 REICHENBACH ON CONGRUENCE

According to Reichenbach[1], all geometrical propositions about physical space describe "a relation between the universe and rigid rods." Physical spacetime can be said to have a certain metrical geometry only relative to some coordinative definition of 'congruent'. The latter involves the correlation of the term 'congruent' with an arbitrary specification of the coincidence behavior of physical congruence standards (such as rigid rods) under transport. We may freely stipulate either that the measuring rods retain their length, or that they change their length in some specified way under transport. Apart from considerations of convenience, the choice of such coordinative definitions is completely arbitrary, since no congruence relation or quantity of length exists independent of and prior to that choice, and relative to which, such correlations could be said to be true or false. Since there is no true congruence relation apart from the behavior of physical measurement standards -- that is, all such congruence relations are specified relative to different choices of coordinating definitions -- geometrical propositions about physical space and time or spacetime do not ascribe genuine properties pertaining to spacetime as such, but are about relations between spacetime points on the one hand, and the coincidence behavior of metrical standards under transport on the other.

Reichenbach[2] illustrates how physical geometry is a function of our interpretation of 'congruent' in terms of the specification of the coincidence behavior of congruence standards under transport. He considers two surfaces, a plane surface with a hemispherical "hump" and a plane surface lying beneath it. We are asked to imagine what it would be like for two-dimensional creatures on each surface to determine by means of two-dimensional rods the geometry of their respective surfaces. Suppose that the hemisphere-creatures measure length in the "normal" way by using rigid rods. They will thus determine a unique congruence class of intervals on their surface and will of course discover that their surface is non-Euclidean. If the plane-creatures were also to use rigid rods they would determine their surface geometry to be Euclidean. But suppose the plane-creatures change their coordinative definition in such a way that two intervals are congruent if and only if they correspond to the projection of the congruent intervals on the hemisphere. Under such an interpretation of 'congruent', which corresponds to a new specification of the coincidence behavior of the rods under transport, the plane-creatures would ascribe a non-Euclidean geometry to their surface, namely, the geometry of the hemisphere which the hemisphere-creatures discovered by measuring length in the "normal" way.

Since the geometry of spacetime depends directly on the metric of spacetime, it follows that the geometry we ascribe to physical spacetime depends directly on the interpretation of 'congruent' and hence on our corresponding method for measuring length. But the crucial question is whether Reichenbach's additional claim is correct, namely, the claim that the interpretation of 'congruent' is conventional, and that there is no fact of the matter with which to arbitrate between different interpretations. Why should we think that the interpretation of 'congruent', and hence the corresponding coincidence behavior of metrical standards under transport, are up for grabs in the way that Reichenbach suggests?

According to Reichenbach, any specification of the coincidence behavior of congruence standards, such as rigid rods under transport, is unverifiable in principle. For example, if the plane people make measurements in the "normal" way, they will assume that the coincidence behavior of congruence standards corresponds to rigid rods which retain their length under transport. But can one prove that this assumption is justified? Reichenbach says no. The plane people may postulate an undetectable "universal force" responsible for expansion and contraction that affects all bodies in the same way. Any proof of the statement that the measuring rods stay the same length will necessarily be viciously circular. No description of such coincidence behavior of congruence standards under transport has an objective truth value; any such description merely constitutes an arbitrary coordinative definition. We may thus have two empirically equivalent theories, one postulating a Euclidean, the other a non-Euclidean geometry. By the principle of verifiability, these two empirically indistinguishable theories are fully equivalent. Are we to conclude that Reichenbach is merely making an epistemological point when he argues for the existence of alternative geometrical descriptions? The answer must be in the negative. Reichenbach's views concerning the nature of geometric structure have a definite ontological flavor. Reichenbach is not only committed to the view that if alternative geometrical descriptions are truly equivalent then they describe the same aspect of physical reality, but also to the full reductionist view, according to which, strictly speaking, there is no such thing as spatial, temporal or spatio-temporal structure independently of and hence prior to some adoption of congruence definition or method of measurement. Rather, spatial and temporal or spatio-temporal geometrical relations are fully reducible to the coincidence behavior of some congruence standards under transport. Nor do forces and force fields enjoy the full-fledged status of membership in a physical ontology of physically real entities whose existence is independent of coordinative definitions of 'congruent'.

> We can define such forces as equal to zero because a force is no absolute datum. When does a force exist? By force we understand something which is responsible for a geometrical change. . . . The existence of force is therefore dependent on the coordinative definition of geometry.[3]

Although Reichenbach adopts a verifiability criterion of equivalence, he interprets it as a demarcation principle to distinguish not only between what can and what cannot be known about physical reality, but also between what can and what cannot be meaningfully said about physical reality. Thus to assert, as does Weyl, that physical spacetime actually has a determinate metric structure and hence a determinate physical geometry, would, from Reichenbach's point of view, lack any possible justification. Not only would Weyl's claim be inadequate epistemically, but such a claim would constitute according to Reichenbach, a completely unwarranted assertion about physical reality itself, and not merely our knowledge of it.

Reichenbach speaks of the set of admissable systems of "geometry and physics" as alternative but equivalent descriptions. The set of all systems of "geometry and physics" that are equivalent to an admissable system (G+P), is an equivalence class of (G+P). According to Reichenbach there is one and only one true equivalence class of "geometry plus physics." Which of the possible equivalence classes is the true one is not a matter of convention. But given the true equivalence class, the choice of one of its members is purely conventional. If two systems (G+P) and (G'+P') of "geometry and physics," are truly equivalent descriptions, then they are either both true or both false; that is, they are either both members of the true equivalence class or they are not. Hence if, as in Reichenbach's example, G represents the Euclidean geometry involved in the description (G+P) and G' represents the non-Euclidean geometry involved in the description (G'+P'), then neither G nor G' represent a geometry of spacetime <u>simpliciter</u>.* The geometry of spacetime is always relative to a choice of the coincidence behavior of congruence standards under transport. Without coordinating definitions of 'congruent', that is, without the adoption of some measuring procedure, physical geometry is not a matter of empirical fact, known or unknown. Once a coordinating definition of 'congruence' has been chosen, the geometry of physical spacetime becomes a matter of empirical fact. In the presence of some suitable coordinating definition of congruence physical geometry is fully determinate and non-conventional.

3.4 GRÜNBAUM ON CONGRUENCE

Grünbaum's version of ontological conventionalism is well known by now. It is based on inductively supported structural considerations concerning the nature of spacetime.

Grünbaum[4] contrasts the allegedly metric amorphousness of a continuous homogeneous manifold with the metric determinateness of discrete manifolds whose intervals are composed of ultimate indivisible elements. He suggests that in a discrete manifold the comparison of non-coincident intervals with respect to their relative size is possible in an intrinsic way by counting the number of quanta in the respective intervals. Two non-coincident intervals are said to be congruent if each contains the same number of quanta. Congruence relations are now definable in an intrinsic way, since we only need to appeal to the intrinsic structural features of the manifold regions themselves in order to determine their relative size.

Grünbaum argues that the structure of a discrete manifold permits an intrinsic definability of congruence relations, but that the structural features of a continuous homogeneous manifold are such that congruence relations cannot be so defined and must be stipulated by means of external congruence standards. A continuous homogeneous manifold does not provide us with an intrinsic structural basis, that is, a basis pertaining solely to the point-field of the manifold itself, by which to judge whether two non-coincident intervals are or are not congruent. If one interval is contained by another, then the diadic predicates "greater than" or "smaller than" express a determinate qualitative relationship between the intervals which constitutes an intrinsic-to-the-manifold fact. But in the case of two non-coincident intervals there exists no intrinsic-to-the-manifold fact by which we can determine their relative size either quantitatively or qualitatively. We can only judge their relative size "extrinsically" by relating or comparing the intervals by means of some "external" congruence standard such as rigid rods.

A discrete space possesses cardinality-based, non-trivial metrics intrinsic to the respective intervals. Thus two intervals are congruent if and only if they have the same cardinality, that is, if they contain the same number of points. Hence a discrete space admits non-trivial congruence classes of intervals generated by a non-trivial metric intrinsic to the respective intervals. In this case, the intension of 'x is congruent to y', given by the cardinality based measure, uniquely determines that congruence class as its extension, since this cardinality-based congruence class so determined is the only class generated by a non-trivial metric intrinsic to the intervals of space.

On the other hand, if a manifold is dense and countable then every interval will have exactly as many points as any other. That is, the sets of points of any interval will all have the same cardinality and will all be of the same order type as the set of rational numbers.

Therefore, the intrinsic point structure will be inadequate in defining congruence relations between two non-coincident intervals. Moreover, if a manifold is continuous, then each interval will consist of a set of uncountably infinite number of points, all intervals now being of the order type of real number intervals.

Any congruence relations or metrical properties of a manifold that depend on the coincidence behavior of extrinsic congruence standards under transport, and hence on states of affairs which do not derive from the manifold structure itself but hold as a matter of fact other than those pertaining solely to the manifold, do not constitute genuine properties of the manifold. Such properties or congruence relations are convention-laden since congruence definitions based on the coincidence behavior of extrinsic congruence standards under transport are not constrained by the manifold structure itself. Hence alternative congruence definitions based on alternative coincidence behaviors of extrinsic congruence standards under transport do not provide an equivalent way of describing one and the same intrinsic-to-the-manifold facts, but constitute alternative ways of imposing congruence relations on the manifold, which, independently of such ascriptions, is metrically amorphous.

On the basis of the metrically amorphous character of the spacetime manifold, the conventional specifications of the coincidence behavior of rigid rods can have no determinate truth value according to Grünbaum, because there are no determinate intrinsic congruence relations relative to which they could be so verified. Thus the sense of conventionality in Grünbaum's case rests on ontological considerations pertaining to the nature of the spacetime manifold itself.

To crystalize the difference between Grünbaum and Reichenbach, consider Reichenbach's discussion of temporal congruence. Temporal congruence relations are functions of some periodic processes which, if uniform, determine that the durations of any two periods are equal. But how can one establish the uniformity of periodic processes? One might answer that one can establish uniformity on the basis of well confirmed physical theory. After all, not every temporal congruence standard will do. Taking the king's heart beat as our temporal congruence standard would mean that whenever the king runs upstairs all natural processes would slow down. Reichenbach expresses this problem as follows:

> Do not the laws of physics, for instance those of the motion of a pendulum, compel us to believe in the equality of the periods? It is true that the laws as described in textbooks suggest this belief; but if we ask ourselves where these laws come from, we shall find that they are obtained through observations of clocks calibrated according to the principle of the equality of their periods. The proof is therefore circular. If we had used a different scale for our measurements, we would have obtained different laws which in turn would have compelled us to consider the latter scale as the correct one.[5]

Thus, according to Reichenbach, the uniformity assumption of temporal processes cannot select one temporal measuring procedure over another since it is itself subject to confirmation by measurements that are themselves obtained by the very procedure in question.

Reichenbach's conventionalism depends upon the assertion of the impossibility of verifying that our congruence standards and the theories which employ them are correct. The possibility of alternative congruence relations via alternative coordinate definitions, and the corresponding alternative theories that are equally supported by evidence and are hence empirically indistinguishable, is what underlies Reichenbach's brand of conventionalism.

But for Grünbaum the possibility of alternative metrizations is not a sufficient condition of the conventionality of congruence relations. Although discrete spaces are not intrinsically metrically amorphous, they do admit, Grünbaum tells us, alternative metrizations in terms of the intrinsic metric structure. But such alternatives congruence relations are not conventional for Grünbaum.

3.5 A CRITIQUE OF GRÜNBAUM'S INTERPRETATION OF RIEMANN

We saw that in Part I of his inaugural lecture, Riemann sets himself the task to work out the general concept of a manifold. Taking the topological structure as essential and basic, Riemann's great philosophical contribution consisted in separating the manifold structure from additional structures (eg., the metric structures) by showing that the concept of a continuous manifold does not by itself involve the specification of a particular metrical relationship. Unlike the case of a discrete manifold, where the determination of a set necessarily implies the determination of its quantity or cardinal number, in the case of a continuous manifold, the concept of such a manifold and of its continuity properties can be separated from its metrical structure. It is in the very beginning of Section I (Part I) where Riemann introduces what I have referred to as Riemann's Conceptual Separation Thesis:

> Particular portions of a manifold, distinguished by a mark or by a boundary, are called quanta. Their quantitative comparison is effected in the case of discrete magnitudes by counting, in the case of continuous magnitudes by measurement. Measuring involves the superposition of the magnitudes to be compared; it therefore requires a means of transporting one magnitude to be used as a standard for the other. Otherwise, one can compare two magnitudes only when one is a part of the other, and then only as to "more" or "less", not as to "how much".[6]

Grünbaum has seized upon the above statement as support for his claim that Riemann's view on the epistemological status of physical geometry is the immediate progenitor of his thesis of the metric amorphousness of continuous space. According to Grünbaum, Riemann is not making an epistemological but an ontological assertion in the above cited passage. He attributes to Riemann two senses of 'relational' for the metric structure of space. The first sense, what Grünbaum labels pre-dynamical thesis of relationality of the metric structure, consists in Riemann's allegedly explicit denial of the received Newtonian view that unique metric interval-ratios exist in continuous space and time independently of any relation to anything external. This, according to Grünbaum[7], Riemann expressed in the above passage by saying that in a continuous space measurement requires some "means of transporting one magnitude to be used as a standard for the other . . .". The second sense of "relational" pertains to Riemann's suggestion at the end of his inaugural lecture that the metrical structure of a continuous manifold is relational in the sense of being dynamically dependent on the matter distribution, or to use Riemann's own words: ". . . the basis of metric relations must be sought for outside it, in binding forces acting upon it."[8]

According to Grünbaum the second sense of 'relational' requires and is dependent on the first. That is for Grünbaum

> . . . some external, transported metric standards or other (eg., rigid rods) function indispensibly as mediating causal agents, rather than as mere manifesting devices, in the dynamical dependence of

> the geometry as such on the matter distribution. In other words, this causal mediation claim asserts that a physical field produced by the matter distribution acquires geometrical significance only via its action on particular kinds of external transported physical standards, which, to begin with, generate metric interval ratios in the continuous spatial manifold altogether.[9]

Hence Grünbaum interprets Riemann to be saying that determinate metric ratios are first generated ontologically in continuous space by external, transported standards, and that the matter distribution can affect the metric geometry dynamically only through the causal mediation of external metric standards. Consequently, in Grünbaum's view Riemann holds that without an external standard there can be neither a non-dynamical nor a dynamical geometry in continuous physical space, so that any dynamicity of the geometry must be mediated by the transported standard.

I shall now argue that this interpretation of Riemann is unsupported by his writings. Riemann nowhere so much as hinted that external congruence standards have an ontologically constitutive role for the metric ratios of physical space. There simply is no evidence to be had in Riemann's writings which would suggest that he endorsed Grünbaum's view that the coincidence behavior of external congruence standards under transport provides an ontological basis for or is ontologically constitutive of the congruence class of physical space or spacetime.

Concerning the above quotation from Riemann's lecture, Grünbaum observes correctly that Riemann takes it as given whether the spatial manifold is continuous or discrete. Riemann takes the topological character of space as given and essential. Now Grünbaum further remarks with respect to the above quotation:

> If physical space is a continuous manifold, he [Riemann, H. K] tells us that -- ontologically and hence epistemically -- 'for measurement there is requisite some means of carrying forward one magnitude as a measure for the other'. And 'in default of this' external transported metric standard, metric comparability of spatial intervals degenerates into a system of merely qualitative relations of proper inclusion. This is clearly a relational conception of metric interval-ratios in the nontrivial sense that determinate metric interval-ratios are held to depend ontologically on an external transported standard in addition to being trivially relational qua being ratios.[10]

I am simply unable to see by what logic of exegesis Grünbaum is able to interpret this passage in Riemann's lecture as producing some evidence for the claim that Riemann assigned an ontological constitutive role to the metric standards. Where does Riemann tell us, or to be more charitable to Grünbaum, where does Riemann provide at least some clues that " -- ontologically and hence epistemically -- for measurement there is requisite some means of carrying forward one magnitude as a measure for the other"? Where is the evidence for Grünbaum's claim "that determinate metric interval-ratios are held to depend ontologically on external transported standards"?

I think it is quite clear from Riemann's inaugural address that Riemann merely wishes to make a conceptual point in the above passage[11]: The possible non-topological, or better still, post-topological structures such as metric relations of which a continuous manifold is susceptible cannot be deduced from the manifold structure. Rather, they must be discovered through experience. It is therefore in the very beginning of the "Plan of the Investigation" that Riemann emphasizes that the general concept of an n-dimensional manifold, of which physical space is a special case, needs to be worked out. It will turn out, Riemann suggests, that an n-dimensional continuous manifold is capable of admitting various metrical relations, and that in the special case of physical space, physical geometry cannot be deduced from the topological structure of space. Rather, Riemann says "those properties through which space [physical three-dimensional space, H. K.] distinguishes itself from other conceivable triply extended magnitudes can only be taken from experience" (my emphasis).[12] And now Riemann raises the problem of how to choose from the possible simple empirical means a system of simple facts from which (aus denen sich) to determine the metrical relations of space; that is, from which to determine those post-topological features of physical space which single the latter out qua physical space. Riemann says:

> Hieraus entsteht die Aufgabe, die einfachsten Thatsachen aufzusuchen, aus denen sich [my emphasis] die Massverhältnisse des Raumes bestimmen lassen -- eine Aufgabe, die der Natur der Sache nach nicht völlig bestimmt ist; denn es lassen sich mehrere Systeme einfacher Thatsachen angeben,

welche zur Bestimmung der Massverhältnisse des Raumes hinreichen; am wichtigsten ist für den gegenwärtigen Zweck das von Euklid zugrunde gelegte.[13]

(Thus arises the problem of seeking out the simplest data from which the metric relations of Space can be determined, a problem which by its very nature is not completely determined, for there may be several systems of simple data which suffice to determine the metric relations of Space;—for the present purpose, the most important system is that laid down as a foundation of geometry by Euclid.)

The phrase "aus denen sich" employed in the first sentence of the above quotation suggests that the "system of simple facts" plays an epistemic role in the determination of geometric structures. Had Riemann written

(a) welche die Massverhältnisse des Raumes bestimmen

or (b) durch welche die Massverhältnisse des Raumes bestimmt werden

instead of

(c) . . . aus denen sich die Massverhältnisse des Raumes bestimmen lassen,

then it would be possible to interpret the expressions (a) and (b), if taken out of context, as attributing to the empirical means either an ontological or an epistemological role. That is, out of context the above modified statements (a) and (b) are neutral with respect to either interpretation. But when read within the overall context of Riemann's discussion, even the modified versions assert that the empirical means used for the determination of the metrical relationships only function as epistemic vehicles. That Riemann used the expression "aus denen sich" (in (c)) clearly indicates that he wished to emphasize that the "empirical means" play an epistemic role only.

In Section 2.3, I pointed out that for Helmholtz our knowledge of ordinary middle sized geometrical objects rests on the fact (Thatsache) of the free mobility of rigid bodies, whereas for Riemann, who speaks of Hypothesen in this context, the concept of rigid bodies may break down in the small. By conceiving solidity as an equilibrium of atomic configuration, Riemann entertains the possibility that the geometric structure in the small is bound up with dynamical considerations. The possible empirical sources that bear on our knowledge of geometry may thus have to be understood as being intimately linked to physical theory. Therefore, Riemann's suggestion that there are several "systems of facts" (Thatsachen) which are sufficient for the determination of the metric structure of space, and that these "facts" are not necessary but are only of empirical certitude, clearly indicates that he is emphasizing their epistemic and not their ontological role. That the "Thatsachen" are not necessary, but are only of empirical certitude, that they are hypotheses, can only mean that they are not necessary because they are only of empirical certitude with respect to their epistemic adequacy in revealing the metrical structure which physical space actually has.

FOOTNOTES

1. [4], [5], [6] and [7].
2. [4], pp. 10-14.
3. [4], p. 27.
4. [5].
5. [4], p. 116.
6. [8], p. 274.
7. [3], p. 318.
8. [8], p. 286.
9. [3], p. 315.
10. [3], p. 316.
11. See also [1], pp. 263-272.
12. [8], p. 273.
13. [8], p. 273.

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CHAPTER 4

RIEMANN AND WEYL'S GEOMETRICAL REALISM

4.1 INTRODUCTION

Grünbaum has repeatedly argued that Weyl's interpretation of Riemann accords with his.[1] I shall show that Weyl could not possibly be in agreement with Grünbaum. An examination of Weyl's philosophy of geometry shows him to be one of the most ardent defenders of geometric realism. That Weyl's understanding of Riemann is the direct opposite of that of Grünbaum's is underscored by the fact that Weyl considers Riemann's work to be essentially progenitorial of his own version of geometric realism.

Since Weyl's mathematical work is in important respects closely related to his philosophical views, a brief outline of his construction of pure infinitesimal geometry and his group-theoretical characterization of the metric structure will serve to facilitate a better understanding of the last remark.

Weyl's group-theoretical characterization of the metric represents his effort to prove that among all the possible metrics that can be put on a continuous manifold, the Pythagorean-Riemannian metric is in important respects unique. An examination of Weyl's group-theoretical analysis will, in particular, provide for a clear appreciation of Weyl's understanding of the historical and modern significance of <u>Riemann's Dynamical Hypothesis</u> and bring in to sharp focus the fundamental difference between himself and Grünbaum.

The motivation for offering a brief outline of Weyl's construction of infinitesimal geometry is twofold. First, it will provide some of the background needed in subsequent chapters. Also, it will show the fruitfulness of treating the affine, projective and conformal geometries in their own right, rather than as mere aspects of the metric geometry. It was, in part, this aspect of Weyl's approach which led to his unified field theory, and also suggested to him the geodesic method by which to discover the metric of spacetime.

All of tensor analysis is based exclusively on the basic affine concept of infinitesimal parallel displacement. No metrical considerations are involved here. Moreover, the important geometric notion of a straight or geodesic line only involves the notion of an affine connection. A line is geodesic when the tangent direction experiences auto-parallel displacement when moved along the line. The process of auto-parallelism of direction is characteristic for the projective structure. The latter arises from affine geometry by abstraction.

The equation $ds^2 = 0$ characterizes the conformal structure of a metric space and constitutes an abstraction from the latter. Weyl suggested that $ds^2 = 0$ should be taken as the starting point. The ratios of the ten coefficients g_{ij} and not their actual values should be taken as determined by experience. This means that length transfer may be non-integrable (path-dependent).

On the basis of these considerations Weyl developed a geometry more general than the Riemannian geometry which had been adopted by Einstein. Within the context of space-time, Weyl's generalized pseudo-Riemannian geometry is not merely specified by the quadratic differential form $ds^2 = g_{ij}dx^i dx^j$, but involves, in addition, a linear differential form $\theta_i dx^i$. The g_{ij} may be interpreted as the potentials of gravitation as in Einstein's theory, whereas the four coefficients θ_i were interpreted by Weyl as the four components of the electromagnetic potential vector. Weyl thought that he had succeeded in a unification of electromagnetism and gravitation by expressing both gravity and electricity as effects of the metric structure of spacetime. With the advent of Quantum Theory however, Weyl had to abandon his unified field theory.

Weyl considered the affine and metric structures as fundamental from a geometric point of view, and saw the projective and conformal structures as arising from them by abstraction. However, he emphasized the fundamental rôles of the conformal and projective structures from a physical point of view. Weyl seems to have been the first to introduce the geodesic or causal-inertial method by suggesting that the conformal structure represents the causal structure, and may be identified with the propagation of light, and that the projective structure represents the inertial structure of spacetime which is revealed by the path structure of the free(fall)motions of suitable test particles.

4.2 WEYL'S FIELD ONTOLOGY OF GEOMETRIC STRUCTURE

The central thesis common to all versions of geometric conventionalism is that we may freely stipulate the geometry of spacetime. There is no such thing as the geometry of spacetime awaiting our discovery.

In sharp contrast, the central thesis of Weyl's geometrical realism is that we are not free to stipulate a geometry. There is such a thing as the geometry of spacetime awaiting our discovery.

I shall characterize Weyl's realism as a field ontology of geometric structure. According to Weyl, the metrical structure of physical spacetime does not depend relationally on the choice of congruence standards, rather, the extension or reference of the congruence class of physical spacetime is already fully determined by the inertial-gravitational or metric field. The metric field is, like the electromagnetic field, a physical field in its own right; it determines the metric ratios of spacetime intervals prior to and independent of any stipulation of external congruence standards under transport.

For Grünbaum on the other hand, the inertial-gravitational field, although a physical field in its own right, acquires geometrical significance only via its action on particular kinds of external standards of congruence. It is the latter, which generate metric interval ratios and not the inertial-gravitational field. We saw that Grünbaum seizes upon certain passages in Riemann's inaugural address in support of the claim that Riemann's work must be seen as the immediate progenitor of his brand of geometric conventionalism. He attributes to Riemann two senses of 'relational' for the metric structure of space. The first sense Grünbaum calls "Riemann's pre-dynamical thesis of relationality" of the metric structure. According to Grünbaum, Riemann expressed this sense of 'relationality', by saying that in a continuous space measurement requires some "means of transporting one magnitude to be used as a standard for the other . . .".

It is quite clear from Weyl's writing, however, that far from being supportive of Grünbaum's conventionalist interpretation of Riemann, Weyl understands Riemann to be merely making a conceptual point in the passage containing the above remark. According to Weyl, Riemann does not espouse a pre-dynamical thesis of relationality of the metric structure, but, as I showed earlier, Riemann wishes to clarify the distinction between assumptions that are logical consequences of any manifold, and those assumptions which restrict the range of structural possibilities implied by the manifold to the special case of physical space: The post-topological structures, such as for example, the metric and affine relations, of which a continuous manifold is susceptible, cannot be deduced from the manifold structure.

Riemann searched for the most general type of an n-dimensional manifold. On this manifold, Euclidean geometry turns out to be a special case resulting from a certain form of the metric. Weyl takes this general structure, the manifold structure, which has certain continuity and order properties, as the space of pure intuition, but leaves the choice of the other geometrical structures, such as the projective, conformal, affine and metric structures, open. Only the spatio-temporal neighborhood has a concrete meaning in intuition.

For Riemann and Weyl, the metrical axioms are no longer dictated, as they were for Kant, by pure intuition. According to Weyl, the metric is not, as it was for Kant, "part of the static homogeneous form of phenomena, but of their ever changing material content".[2] Weyl says in this context:

> The conceptual separation of its structure from the underlying amorphous continuum, the recognition that space as such is merely the medium of 'contact', is already indicated in the Aristotelian idea of space.[3]

We saw that Riemann assumed the validity of the Pythagorean metric only in the infinitely small, and that his geometric assumptions rest on infinitesimally small magnitudes. Being essentially a geometry of infinitely near points it conforms to the Leibnizian idea of the continuity principle: All laws are to be formulated as field laws and not by action-at-a-distance. Weyl says:

> Inspired by Gauss's theory of curved surfaces, Riemann assumed that Euclidean geometry holds in the infinitely small . . . As the true lawfulness of nature, according to Leibniz's continuity principle, finds its expression in laws of

> nearby action, connecting only the value of physical quantities at spacetime points in the immediate vicinity of one another, so the basic relations of geometry should concern only infinitely closely adjacent points ('near-geometry' as opposed to 'far-geometry'). Only in the infinitely small may we expect to encounter the elementary and uniform laws, hence the world must be comprehended through its behavior in the infinitely small.[4]

Hence according to Weyl, just as Faraday's field interpretation of electromagnetic phenomena may be contrasted with an action-at-a-distance interpretation, so Riemann's infinitesimal geometric standpoint may be contrasted with "distance-geometry", namely, Euclidean and ordinary non-Euclidean geometry. Weyl remarks:

> We differentiate now between the amorphous continuum and its metrical structure. The first has retained its a priori character . . . whereas the structural field (Strukturfeld) is completely subjected to the powerplay of the world; being a real entity, Einstein prefers to call it the ether.[5]

Hence by formally separating the post-topological structures such as the affine, projective, conformal and metric structures from the manifold, so that these structures are no longer rigidly tied to it, (Riemann's Conceptual Separation Thesis), Riemann deprived them of their formal geometric rigidity and, on the basis of his infinitesimal geometric standpoint or "near-geometry", allowed for the possibility to interpret them as mathematical representations of flexible, dynamical physical structural fields (Strukturfelder) on the manifold of spacetime, geometrical fields which reci-

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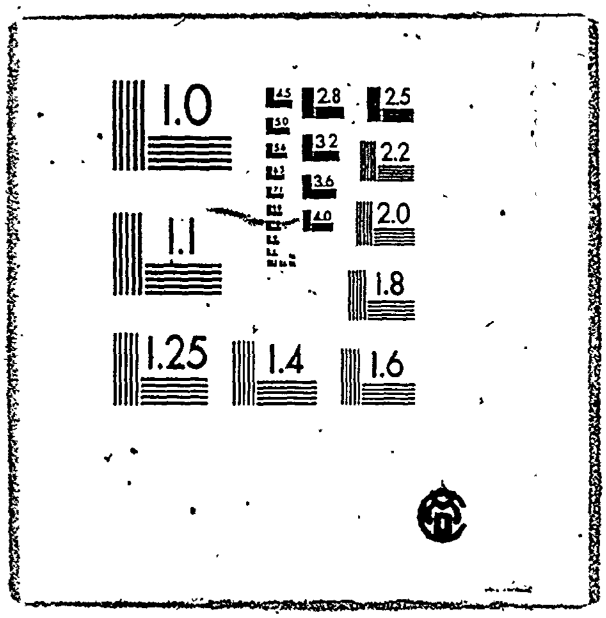

procally interact with matter.

Riemann's recognition that the metric structure should be separated from the manifold structure, together with his adoption of the infinitesimal standpoint, were prerequisite steps for the development of differential geometry as the mathematics of differential geometric fields on manifolds.

When interpreted physically, that is when viewed as mathematical representations of physical spacetime structure, these mathematical structures or geometrical fields correspond, according to Weyl, to physical structural fields. Analogous to the electromagnetic field, these structural fields act on matter and are in turn acted on by matter. Weyl says:

> All our considerations until now were based on the assumption, that the metric structure of space is something that is fixed and given. Riemann already pointed to another possibility which was realized through General Relativity. The metrical structure of the extensive medium of the external world is a field of physical reality, which is causally dependent on the state of matter.[6]

> I now come to the crucial idea of the theory of General Relativity. Whatever exerts as powerful and real effects as does the metric structure of the world, cannot be a rigid and once and for all fixed geometrical structure of the world, but must itself be something real which not only exerts effects on matter but which in turn suffers them through matter. Riemann already suggested for space the idea that the structural field (Strukturfeld), like the electromagnetic field, reciprocally interacts with matter.[7]

> We already explained with the example of inertia, that the structural field (Strukturfeld) must, as a close-action, (Nahewirkung), be understood infinitesimally. How this can occur with the metric structure of space, Riemann abstracted from Gauss's theory of curved surfaces.[8]

Notice, it is not claimed that the various geometrical fields are "intrinsic" (intrinsic in Grünbaum's sense) to the manifold structure M of spacetime. M represents an amorphous four-dimensional differentiable continuum in the sense of analysis situs and has no properties besides those that fall under the concept of a manifold. Weyl says:

> The metric is not a property of the world in itself, rather spacetime as a form of appearance is a completely formless four-dimensional continuum in the sense of analysis situs, but the metric expresses something real, something which exists in the world, which exerts through centrifugal and gravitational forces physical effects on matter, and whose state is conversely conditioned through the distribution and nature of matter.[9]

The amorphous four-dimensional differentiable manifold possesses a high degree of symmetry. Due to its homogeneity, all points are alike; there are no objective geometric properties that enable one to distinguish one point from another.

The full homogeneity or symmetry of space must be described by its group of automorphisms. Those one-to-one mappings of the point field onto itself which leave all relations of objective significance between points undisturbed, form a group, the group of automorphisms. If a geometric object F, that is a point set with a definite relational structure is given, then those automorphisms of space which leave F invariant, constitute a group and this group describes exactly the symmetry which F possesses. For instance, to use an example by Weyl, if $R(p_1, p_2, p_3)$ is a ternary relation that asserts p_1, p_2, p_3 lie on a straight line, then we require that any three points satisfying this relation R are mapped by an automorphism into three other points p_1', p_2', p_3' fulfilling the same relation.

The group of automorphisms in geometry describes as it were, the measure of homogeneity or symmetry of space.

The group of automorphisms of the n-dimensional number space consists of the identity alone. The reason for the non-existence of automorphisms other than the identity is due to the fact that all numbers are distinct individuals. They constitute objects with individuating properties. It is essentially for this reason that the real numbers are used for coordinate descriptions. Whereas the continuum of real numbers consists of individuals, the continua of space, time and spacetime are homogeneous. Spacetime points do not admit of an absolute characterization; they can be distinguished only by "a demonstrative act, by pointing and saying here-now".

It lies in the nature of space and time as the forms of appearances, that they are homogeneous in the sense just described. Weyl says:

> Intuitive space and intuitive time are hardly the adequate medium in which physics is to construct the external world.

> No less than the sense qualities must the intuition of space and time be relinquished as its building material; they must be replaced by a four-dimensional continuum in the abstract arithmetical sense.[10]

By mapping a given spacetime homeomorphically onto the real number space, on, what Weyl calls, a qualitatively non-differentiated field of free possibilities -- the continuum of all possible coincidences -- we represent spacetime points by their coordinates corresponding to some coordinate system. Thus, spacetime constitutes a four-dimensional continuum, which for its mathematical representation must be, to use an expression of Weyl, spun over by a division net. The four-dimensional arithmetical space can be utilized as a four-dimensional schema for the localization of events of all possible "here-nows".

Physical dynamical quantities in spacetime, such as the geometrical structural fields on the four-dimensional spacetime continuum are describable as functions of a variable point which ranges over the four-dimensional number space $\mathbb{R}^4$.

Instead of thinking of the spacetime points as real substantival entities and any talk of fields as just a convenient way of describing geometrical relations between them, we think of the geometrical fields, such as the projective, conformal (causal) and metric field, as real physical entities with dynamical properties, such as, energy, momentum and angular momentum, and the field points as mere mathematical abstractions. Spacetime is not a medium in the sense of the old ether concept. No ether in that sense exists here. Just as the electromagnetic fields are not states of a medium but constitute independent realities which are not reducible to anything else, so the geometrical fields are independent irreducible physical fields.

> Magnitudes, such as the temperature of a body, or the field strength of the electromagnetic field which have at each spacetime point a definite value, appear as functions of four variables, namely the spacetime coordinates. In a similar fashion, the quantitative course of the metric field obeys exact natural laws; namely Einstein's laws of gravitation, which resemble the Maxwell laws of the electromagnetic field.[11]

In describing the structural field it is necessary to specify a group of transformations, in the tangent space $T(M_p)$ of each point in spacetime. From a geometrical and physical point of view, the invariants under this group of the physical geometrical fields, provide a complete description of the fields. In relativity theory, this group is isomorphic to the Lorentz group and hence leaves invariant a pseudo-Riemannian metric of Lorentzian signature.

An n-dimensional manifold M whose only properties are those that fall under the concept of a manifold, Weyl interprets physically as an n-dimensional empty world, that is, a world empty of both matter and fields. Similarly, an n-dimensional manifold M which is an affinely connected manifold Weyl interprets physically as an n-dimensional world filled with a gravitational field. And an n-dimensional manifold M endowed with a projective structure represents an n-dimensional non-empty world filled with an inertial-gravitational field, or what Weyl calls the guiding field. In a similar vain, an n-dimensional manifold M which possesses a conformal structure of Lorentz type, represents a non-empty n-dimensional world filled with a causal field. And finally, an n-dimensional manifold M endowed with a metrical structure may be interpreted physically as an n-dimensional non-empty world filled with a metric field.

All of these structural fields may from a modern mathematical point of view be represented as cross sections of an appropriate fiber bundle over the manifold M.[12] That is, the amorphous manifold M has associated with it various geometric fields in terms of a mapping of a certain kind from the manifold M to the appropriate bundle over M. For example, a vector field on M is a mapping of a certain kind from the manifold M to its vector bundle. That is, a vector field on M is a cross-section of the vector bundle over M, and a tensor field is a cross-section of the tensor bundle over M. In particular, Einstein's General Theory of Relativity postulates a physical field, the metrical field, which, mathematically speaking, may be characterized as a certain mapping of M to the bundle of non-degenerate, second order symmetric covariant tensors of Lorentz signature over M.

Thus when we say with Weyl that the physical manifold M of a spacetime theory carries a metrical field, we merely express Riemann's ideas in modern terms by saying that the physical metrical field exists on M in the sense that is mathematically characterizable as a certain mapping from M to the corresponding bundle over M.

Einstein's recognition that gravity is not a force-field existing besides a rigid inertia-determining world-geometry, but that gravity should itself be considered as part of the metrical and affine structure of spacetime, helped Riemann's ideas to victory by transforming the structural fields from absolute entities rigidly tied to the spacetime manifold, into dynamical geometrical structural fields on M that reciprocally interact with matter.

It is clear that the views of Riemann, Helmholtz and Weyl are directly opposed to those of geometrical conventionalism. According to Weyl, the four-dimensional continuum is not amorphous but "carries" a structure. Take for example the world of Newtonian spacetime. It is stratified by hyperplanes of absolute simultaneity and in addition possesses a fiber structure consisting of worldlines that intersect these planes. All simultaneous world points on a particular hypersurface (t = constant) constitute a three-dimensional manifold homeomorphic to Euclidean three-space, and all world points of constant spatial location constitute a one-dimensional submanifold, a fiber associated with a worldline. Weyl says of the world structure:

> However this structure is to be exactly and completely described and whatever its inner ground might be, all laws of nature show that it constitutes the most decisive influence on the process of physical events: The behavior of rigid bodies and

> clocks is almost exclusively determined through the metric structure, as is the pattern of the motion of a force-free mass point and the propagation of a light source. And only through these effects on the concrete natural processes can we recognize this structure.[13]

To emphasize, Weyl is not saying that the metric structure is determined by the specification of the behavior of rigid bodies and clocks. Rather,

> the congruent mappings of space form a group Δ^+ of transformations which we call the group of Euclidean motions . . . The fact suggests an interpretation according to which the group Δ^+ of congruent mappings expresses an intrinsic structure of space itself; a structure stamped by space on all spatial objects.[14]

According to Riemann, Helmholtz and Weyl, we _discover_ through the behavior of physical phenomena an _already determined_ metrical structure of spacetime. The metrical relations of physical objects are determined by a physical field, the metric field, which is represented by the second rank metric tensor field. Contrary to geometric conventionalism, spacetime geometry is not about rigid rods, ideal clocks, light rays or freely falling particles, except in the derivative sense of providing information about the physically real metric field which is as physically real as is the electromagnetic field, and which _determines_ and _explains_ the metrical behavior of congruence standards. The metrical field has physical _and_ metrical significance, a significance which does not consist in the mere articulation of relations obtaining between rigid rods or ideal clocks.

Grünbaum, on the other hand, considers all post-topological structures (eg., metric, affine, projective structures) as extrinsic properties. Of course there is no disagreement here. The metric field, for example, is not intrinsic to the local differential topological structure of the spacetime manifold, but is on the manifold in the sense just described. But according to Grünbaum the inertial-gravitational field, though an irreducible physical field, does not have any geometrical properties. Its metrical property is fully reducible to the relations of the chosen external standards of congruence under transport. Only through prior stipulation of congruence standards does the inertial-gravitational field acquire the status of a geometric or metric field.

Grünbaum is not merely saying that there are no intrinsic properties in the sense that there are no purely differential topological properties that permit the canonical definability of the congruence class of spacetime, but he is making the stronger claim, that no other spacetime properties exist (independent of and prior to stipulated congruence standards) which uniquely determine the extension of 'congruence' and 'geodesic'. He takes Riemann's remark that "in a continuous space the metric must come from somewhere else" to assert not merely the view that there is no intrinsic-to-the-manifold basis with which to single out particular metric relations, (ie., Riemann's Conceptual Separation Thesis), but to assert something stronger, namely the claim that no other properties besides the intrinsic-to-the-manifold properties can canonically determine the reference of 'congruence'. This stronger claim is based on what Grünbaum calls Riemann's Metric Hypothesis (RMH). It asserts the intrinsic metric amorphousness (IMA) of spacetime in the following stronger sense: In a continuous homogeneous space there is no intrinsic-to-the-manifold property that is capable of providing a canonical basis on which to specify a specific metric structure, meaning, there is no differential topological basis relative to which a specific metric can be defined <u>and</u> all post-topological structures are extrinsic and therefore do not qualify to provide an objective, non-arbitrary basis for metric definability.

Clearly, if only the local differential topological properties count as intrinsic-to-the-manifold properties of spacetime and if only those geometric properties are objective that are intrinsic properties in Grünbaum's sense of 'intrinsic', then it follows from the truth of RMH, that no post-topological properties which are not definable in terms of the local differential topological properties of the spacetime manifold can be objective and non-arbitrary.

However, if one does not exclude at the outset the possibility that there exists a non-empty set of physically real post-topological geometrical features of spacetime, which, though not intrinsic in Grünbaum's sense of 'intrinsic', are nonetheless intrinsic geometric properties of spacetime in the sense that they do not depend relationally on the behavior of actual or possible congruence standards, but which physical spacetime has and would have in the latter's absence, then RMH does not constitute an analytic inference from the concept of a continuous homogeneous manifold, but is a synthetic claim about spacetime. RMH is false if there exist in fact objective irreducible features of spacetime, such as the inertial-gravitational field, which uniquely determine the extension of 'congruence'.

It would seem that what Grünbaum means by intrinsic properties are those properties which a manifold has and would have in the absence of any external congruence standards. Now Grünbaum considers the differential topological properties to be intrinsic and tries to establish that the metrical properties cannot be defined intrinsically, that is, in terms of the spacetime manifold's differential topological properties and order relations. But what does that show? It certainly does not show that metrical properties are not objective. It seems that all that Grünbaum has established is that there is a sense of 'intrinsic' relative to which the metrical properties of a continuous manifold are not intrinsic. But this goes no way in showing that the metrical properties are not real objective features of physical spacetime. All that it does show is, that metrical properties are not definable in terms of Grünbaum's sense of 'intrinsic definability'.

The assumption that the local differential topological structure and order relations are objective features of physical spacetime seems to be a quite reasonable one. It is difficult to see how one could do physics without at least this much structure. The very possibility of assigning coordinates presupposes that physical spacetime be such that at least locally, it is homeomorphic to E^4 and that the homeomorphisms satisfy the differential compatibility condition. But the inference from this assumption to the ontological claim, which says that every spacetime property which is not definable in terms of the local differential topology and order relation does not constitute an objective property of physical spacetime, but depends for its existence on some stipulated coincidence behavior of external congruence standards under transport, is clearly a non-sequitor. To sustain this ontological claim by simply asserting that the differential local topological structure is factual and objective in a way that the metrical, affine or projective structures are not, because these structures are conventional whereas the differential topological structure is not, is simply question begging. A property which is objective and factual can at most be conventional in an epistemic but not in an ontological sense. Therefore we cannot know whether metrical properties are ontologically conventional unless we first establish whether or not they are factual features of physical spacetime.

I will now argue in somewhat more detail that Grünbaum has not produced any clear and convincing arguments in support of the pre-dynamical relationality of the metric.

His contention, that the g_{ij}-field only enjoys the derived status as the physical metric tensor field of spacetime -- a further status which he thinks is ontologically mediated by external congruence standards, (OI) -- is, as far as I can make sense of it, theoretically unconvincing and physically implausible.

It is important to note in this connection that Grünbaum now admits what he failed to acknowledge explicitly in his earlier writings, namely, that "DH [Dynamical Hypothesis, H. K.] itself no more logically entails OI than the latter entails the former". He goes on to say:

> Yet it is important to see that DH does not entail OI: the kind of externality inherent in the causal dependence of metric structure on the distribution of the matter immersed in space does not itself require logically that this causal dependence also be mediated constitutively -- as distinct from merely manifested -- by an external, transported metric standard of one kind or another. Instead, taken by itself, DH allows that the dynamically dependent metric structure be merely manifested rather than first generated by such standards. For DH allows that a physical field produced by the matter distribution does not first have to acquire geometrical significance from its action on those transported bodies that count as metric standards.
>
> On the other hand once it is assumed with Riemann that OI is true, then, of course, the matter distribution can affect the metric geometry dynamically only through the then essential causal mediation of external metric standards.[15]

Now apart from whether or not OI is true, I wish to re-emphasize that contrary to Grünbaum's contention, Riemann did not espouse Grünbaum's OI-thesis. I showed in the earlier chapters that there is no textual support for Grünbaum's construal of Riemann that suggests that Riemann held the following view: In a continuous space determinate metric ratios are first generated ontologically by external standards of congruence under transport. Without such congruence standards neither a dynamical nor a non-dynamical geometry is possible in continuous space. Consequently, the dynamical character of geometry requires the mediation of external congruence standards (OI).

Having admitted that DH does not unilaterally entail OI, what are Grünbaum's arguments in support of OI, which constitutes, as he now concedes, a physical hypothesis? That is, why is Weyl's contention, that the physical structural g_{ij}-field determines a pseudo-Riemannian metric throughout spacetime without the ontological mediation of external congruence standards, while not to be excluded on logical grounds, nonetheless an untenable hypothesis? In particular, why should we accept Grünbaum's position that in a non-empty spacetime external congruence standards do play an ontologically constitutive role with respect to its metrical properties such that the gravitational fields of matter-empty spacetimes are devoid of geometrical significance. Why should we give credence to such a position and consider the non-relationalist view as an unacceptable alternative, although its unacceptability cannot be established on logical grounds? Grünbaum says that

> . . . it is the idealized behavior . . . of this particular important class of

> external entities that is taken to be canonical for the metricality of these space-times in the sense that it is fundamentally constitutive ontologically for their metric structure. Thus what is otherwise just a physical tensor field $g_{\mu\nu}$ thereby acquires specifically geometric, additional, physical significance as being the metric tensor of the space-time manifold . . . Hence the physical field $g_{\mu\nu}$ also plays an ontologically constitutive role in the metric structure of space-time, but its role is ontologically a derivative or conditional one: it acquires this role only because it causally governs those external entities which play the more fundamental constitutive role ontologically. Were it not for this geometrically canonical role of photons, atomic clocks, etc., and for the "orchestrating" action of the $g_{\mu\nu}$ field on them, what else would confer on $g_{\mu\nu}$ the additional status of being physically the metric tensor of . . . space-time?[16]

But the question "what else would confer on the g_{ij}-field the additional status of being a metric field" already assumes that the g_{ij}-field is metrically amorphous prior to and independent of the mediary activities of the congruence standards. We saw that Grünbaum's argument concerning the ontological issue of the absoluteness versus relationality of the metric is predicated on his distinction between what is "intrinsic" to the spacetime manifold and what is "external" to it. But this merely establishes, as I have argued, that there is a sense of "intrinsic" relative to which the metrical properties of the g_{ij}-field are not "intrinsic". This does not, however, establish that its metrical structure is relational. All it shows is, that the metrical properties of the g_{ij}-field are not definable in terms of Grünbaum's sense of 'intrinsic definability'. To assume therefore that its metricality is relationally and ontologically dependent on the existence of some external congruence standards by simply asserting that the local differential topological structure is factually definite and nonarbitrary in a way that the geometrical structures of the causal-inertial field (such as the conformal, projective, affine and metrical structures) are not, because whatever structural features it may possess are conventional whereas the local differential topological structure is non-conventional, is question begging.

Moreover, Grünbaum's suggestion in the above citation that the physical g_{ij}-field also plays an ontological role but that such a role is a derivative one and that it requires this derivative role because it constrains the behavior of physical systems whose role is ontologically somehow more fundamental, is clearly rather vague.

One would like to get some clear and definite answers from Grünbaum as to why he is so reluctant to go along with the ordinary standard view, according to which alternative methods and alternative physical probative systems are employed to discover the structural properties of the g_{ij}-field and according to which these properties are presumed to exist independently of being explored by these probative manifestational devices. Grünbaum says:

> Can the conclusion drawn by these rhetorical questions be gainsaid on the basis that the external metric devices

> can properly be said to explore the physical $g_{\mu\nu}$ field by which they are governed, and hence to adjust to it? Photons, atomic clocks, infinitesimal rods, and free massive (gravitationally monopole) particles -- hereafter the "family of concordant external metric standards" or "FCS" -- explore the physical $g_{\mu\nu}$ field in the sense that they exhibit the latter's effects on them by concordantly adjusting to that field. But it does not follow that just because the $g_{\mu\nu}$ field can thus be explored and, qua physical field, does indeed exist independently of being explored, this tensor field must have the particular added physical significance of being the metric tensor field of . . . space-time without the ontologically constitutive mediation of the FCS. In other words, just because the FCS "adjusts itself" to the $g_{\mu\nu}$-field as a result of this field's action upon that family of external standards, it does not follow that the entity to which the FCS does adjust itself must be an autonomously geometric structure in whose metricality the FCS plays no ontologically constitutive role but which the FCS merely manifests by its behavior.[17]

But what are Grünbaum's grounds for saying that it does not follow that the entity to which the FCS does adjust itself must be an autonomously geometric structure, and what does he mean by the concept 'adjustment' in this context? Grünbaum goes on to offer what seems to be a rather vague (and I think a somewhat inappropriate) analogy which contributes little by way of clarification when he says

> . . . to say that a conductor orchestrates the playing of his musicians and that they take directions from him is not to say that the directive motions of his hand alone would constitute a concert.[18]

It is worthwhile to digress for a moment to discuss a notion of 'adjust' or 'adjustment' according to which it is correct to say that rigid rods and atomic clocks fail to fully manifest the basic underlying geometric character of the causal-inertial field by adjusting to this field, that is, by interacting with the field in a certain way.

Let us suppose that spacetime has the structure of a Weyl geometry.[19] Such a spacetime would not only have a curvature of direction (Richtungskrümmung) but would also have a curvature of length (Streckenkrümmung). Because of the latter property the formal characterization of congruence transfer would be non-integrable in such a spacetime.

Generalizing Riemannian geometry by introducing in addition to direction curvature also the notion of length curvature and using the latter to describe the electromagnetic field, Weyl showed that the metric field may possess enough degrees of freedom to account for all the effects which were explained by postulating the electromagnetic field.

In a spacetime that is formally characterizable as a Weyl geometry, an actual non-integrable transfer of congruence would mean that two identical clocks with identical initial periods at some point would no longer coincide in the event of a subsequent encounter (after traveling along different paths) with respect to their periods.

Now such a behavior of atomic clocks does not obtain in reality because atoms emit spectral lines of definite frequencies independent of their histories. The constancy of the ratios of the masses, lifetimes and transition frequencies of nuclei and elementary particles suggest that those atomic units are integrably transported. Hence the gravitational standard time defined by a Weyl geometry does not coincide with atomic time. Now one may do one of several things. One may set Weyl's Streckenkrümmung equal to zero (reduction to a pseudo-Riemannian geometry), or one may explain the constancy of the spectral lines of definite frequency of atomic clocks by some notion of "adjustment", or one may find some other physical reason to account for Weyl's curvature of length.

Weyl suggested the second alternative in order to account for the discrepancy between the actual behavior of atoms and the formal notion of a non-integrable congruence transfer of a Weyl geometry. Weyl explained the discrepancy between the notion of transfer of congruence and the actual behavior of measuring rods and atomic clocks in the following way:

> What is the cause of this discrepancy between the idea of congruent transfer and the behaviour of measuring-rods and clocks? I differentiate between the determination of a magnitude in Nature by "persistence" (<u>Beharrung</u>) and by "adjustment" (<u>Einstellung</u>). I shall make the difference clear by the following illustration: We can give to the axis of a rotating top any arbitrary direction in space. This arbitrary original direction then determines for all time the direction of the axis of the top when left to itself, by means of a <u>tendency of persistence</u> which operates

from moment to moment; the axis experiences at every instant a parallel displacement. The exact opposite is the case for a magnetic needle in a magnetic field. Its direction is determined at each instant independently of the condition of the system at other instants by the fact that, in virtue of its constitution, the system adjusts itself in an unequivocally determined manner to the field in which it is situated. A priori we have no ground for assuming as integrable a transfer which results purely from the tendency of persistence . . . Thus, although, for example, Maxwell's equations demand the conservational equation $de/dt=0$ for the charge e of an electron, we are unable to understand from this fact why an electron, even after an indefinitely long time, always possesses an unaltered charge, and why the same charge e is associated with all electrons. This circumstance shows that the charge is not determined by persistence, but by adjustment, and that there can exist only one state of equilibrium of the negative electricity, to which the corpuscle adjusts itself afresh at every instant. For the same reason we can conclude the same thing for the spectral lines of atoms. The one thing common to atoms emitting the same frequency is their constitution, and not the agreement of their frequencies on the occasion of an encounter in the distant past. Similarly, the length of a measuring-rod is obviously determined by adjustment, for I could not give this measuring-rod in this field-position any other length arbitrarily (say double or treble length) in place of the length which it now possesses, in the manner in which I can at will pre-determine its direction. The theoretical possibility of a determination of length by adjustment is given as a consequence of the world-curvature, which arises from the metrical field according to a complicated mathematical law. As a result of its constitution, the measuring-rod assumes a length which possesses this or that value, in relation to the radius of curvature of the field.[20]

It is to be noted that Weyl's formal construction of a pure infinitesimal geometry does not assert anything about the actual behavior of material congruence standards under transport. Reichenbach charges Weyl incorrectly with a priorism when he says:

> This objection must be made to Weyl's generalization of the theory of relativity which abandons altogether the concept of a definite length for an infinitesimal measuring rod. Such a generalization is possible, but whether it is compatible with reality does not depend on its significance for a general local geometry. Therefore, Weyl's generalization must be investigated from the viewpoint of a physical theory, and only experience can be used for a critical analysis. Physics is not a "geometrical necessity"; whoever asserts this returns to the pre-Kantian point of view where it was a necessity given by reason.[21]

From what Weyl calls the epistemological principle of the relativity of magnitude (Grösse) it does not follow that the actual congruent displacement of length is non-integrable. Weyl's principle merely shows that integrability need not be the case. It suggests that we have no a priori reason for assuming that a congruence transfer which results purely from the tendency of persistence is integrable. If integrability does in fact obtain, then this is a consequence of physical natural law. The natural laws of interaction which are adequate to determine how the field affects the behavior of the physical systems such as rigid bodies and clocks, constitute an essential part of our knowledge of the field. The actual process of material congruence standards must therefore be distinguished from the ideal process of congruent displacement of length which is employed in the mathematical construction of spacetime geometry. Weyl says:

> At first glance it might be surprising that according to the pure close-action geometry length transfer is non-integrable in the presence of an electromagnetic field. Does this not clearly contradict the behavior of rigid bodies and clocks? The behavior of these measurement instruments, however, is a physical process whose course is determined by natural laws and as such has nothing to do with the ideal process of 'congruent displacement of spacetime distance' that we employ in the mathematical construction of the spacetime geometry. The connection between the metric field and the behavior of rigid rods and clocks is already very unclear in Special Relativity theory if one does not restrict one self to quasi-stationary motion. Although these instruments play an indispensable role in praxis, it is clearly incorrect to define the metric field through the data that are directly obtained from these instruments.[22]

And in another place Weyl says:

> If the primordial structure of the ether [synthesis of gravitational and electromagnetic field, H. K.] is of a metric nature, then the tendency of persistence of world directions revealed by a body in free motion must be based on a tendency of persistence of length. If we associate, however, with the isolated masspoint a clock, then we may perhaps discover from it the parallel displacements of directions, but not that of lengths, since the latter are given incorrectly by the readings of the clock being compensated for by the adjustment to the world curvature. Consequently there exists a difference between the primordial geometry of the ether and what is read off the measurement instruments, the so called natural geometry. The second arises from the first by compensating the infinitesimal congruent

displacements of lengths through their
adjustment to curvature.[23]

Now let us suppose for a moment that the world is such that a second clock effect would in fact be realized by atomic clocks when transfered along different paths. Would Grünbaum have us believe then that the resulting path dependency of congruence under transport is a relational property? In such a world two identical clocks α and β, with identical initial periods at p, that travel to q along different paths A and B respectively, would not only exhibit a relativistic time difference (first clock effect) upon their subsequent encounter at q, but they would also be ticking at different rates (second clock effect). How could one even attach any meaning to the claim that, say, clock α will, by virtue of the path it is moving along, be ontologically constitutive of its change of rate in ticking?

It seems that path dependency of congruence transfer is a non-relational property of the causal-inertial field. It is the latter which is causally responsible for the second clock effect, that is the change of rate in ticking of the clocks upon their subsequent encounter after having been transfered along different paths. But if it is reasonable and suggestive to regard the causal-inertial field as being causally responsible for the rate of change in ticking, then why should it not also be regarded as having the character of an autonomous metric field that dynamically determines the proper times which clocks measure along their own world line.

The empirical fact that the actual behavior of atomic clocks and measuring rods do not exhibit non-integrable congruence transfer Weyl explained, as we saw from the above citation, by the notion of adjustment to the curvature of the causal-inertial field. Weyl's remarks here are reminiscent of Riemann. We recall Riemann's suggestion that the possible empirical sources of information which provide epistemic access to the metrical structure of space cannot be treated in isolation from our physical theories. The issue of geometric structure is intimately bound up with our whole understanding of physical interaction. Riemann's prophetic remarks that the concept of solid bodies may break down in the small, suggest that for him geometric structure in the small involves dynamical considerations, that is, it involves taking into account the atomic constitution of matter and hence conceiving of solidity as an equilibrium of atomic configuration. Thus Riemann spoke of Hypothesen in this context to emphasize that our congruence standards are only of empirical adequacy in revealing the metrical structure which physical space actually has.

Of course Grünbaum might still reply that the very notion of the path dependency of congruence transfer presupposes his pre-dynamical relationalist doctrine of congruence, and I strongly suspect that this is what his response would be. Since I really do not fully understand Grünbaum's OI-thesis, I cannot be sure that my discussion counts much against that position. One can, however, say at least this:

Grünbaum admits that OI is not entailed by DH and that OI is a physical hypothesis. Therefore, any self-consistent defence of either Grünbaum's OI-thesis or of Weyl's non-relationalist position ought to satisfy as a minimum some intuitive criterion of theoretical intelligibility and physical plausibility.

It is apparent from the above discussion that Weyl's non-relationalist position concerning the metric structure of spacetime is clear, consistent and physically plausible. On the other hand one gets the impression from Grünbaum's writings that the notions and maneuvers employed in defence of his views are likely to cost more than the relationalist thesis they are called upon to rescue is worth.

The crucial point is whether the causal-inertial field explains or only describes congruence transfer. The world could be such that Weyl's approach to unify gravitation with electromagnetism would have been successful. In that event the nature of Weyl's discovery would be fundamentally distorted if one were to regard the causal-inertial field as merely a pre-geometrical physical field, a descriptive convenience that "orchestrates" the relations holding between measuring rods and clocks. "If our hypothesis is true" Weyl says, "then we do not have to do any more with two independently existing fields that have no inner connections with each other, rather the ether . . . is a (3+1)-dimensional extensive medium of metrical structure"[24] (Weyl's emphasis).

Grünbaum has certainly not established anything like a plausible and hence preferable relationalist alternative to Weyl's non-relationalist field ontology of geometric structure according to which the <u>extension</u> or <u>reference</u> of the congruence class of physical spacetime is already fully determined by the causal-inertial field. Merely elaborating a distinction between topological and metrical structures does not establish that the spatio-temporal metric is non-objective, non-factual. That is, the non-definability of the metric in terms of the manifold topology of spacetime is not sufficient to establish that it is not an objective property of spacetime in the sense that the metric ratios of physical spacetime intervals are fully determined by the inertial-gravitational field prior to and independent of any stipulation of external congruence standards under transport.

4.3 THE MANIFOLD STRUCTURE

Conceived in its broadest sense, differential geometry is the mathematics of differentiable geometric fields on manifolds. We begin, therefore, with some remarks on the concept of a manifold, and characterize the underlying basic structure, i.e., the manifold structure which carries various geometrical fields on it.

The manifold structure of a mathematical model, the four-dimensional schema of the physical spacetime of a spacetime theory, represents the topological and differentiable properties of spacetime. A topological space is a non-empty set with sufficient structure to allow for the definition of neighboring points and continuous functions. The topology of spacetime determines such characteristics as whether or not spacetime is compact and whether or not it is simply connected. The differential structure fixes the maximum degree of smoothness (differentiability) of functions on spacetime.

Let M represent the four-dimensional topological manifold of spacetime which has a countable basis, is Hausdorff, connected, and locally chartable. Instead of the term "locally chartable" it is customary to use the term "locally Euclidean". The latter expression, however, suggests too much structure. That M is locally chartable means only that for every point $p \epsilon M$, there exists a chart (U,x) for M, that is, an open neighborhood U of p and a homeomorphism $x: U \to x_{\vdash}(U)$. $x_{\vdash}(U)$ is an open neighborhood of $\mathbb{R}^4$ and $\bar{\mathbb{R}}^4$ denotes the Cartesian product $\underset{4}{X}\mathbb{R}$ equipped with the product topology and the product differentiable structure. In particular, M does not acquire any of the additional structure of the spaces V^4, O^4, A^4, or $\mathbb{E}^4$, equipped with the structure of a vector space, an inner product space, an affine space and an euclidean space respectively. M does not even inherit a differentiable structure from $\mathbb{R}^4$, unless the family of charts for M satisfies certain compatibility conditions which will now be described.

A <u>C^∞ atlas</u> $\tilde{A}(M)$ for an n-dimensional topological manifold M is a family of charts $\{U_\alpha, x_\alpha\}_{\alpha\varepsilon\tilde{I}}$ such that

$M = \underset{\alpha\varepsilon\tilde{I}}{\cup} U_\alpha$ and

$$\forall\alpha\beta\varepsilon\tilde{I} : U_\alpha \cap U_\beta \neq \emptyset \quad \Lambda_{\alpha\beta} \text{ is } C^\infty .$$

where $\Lambda_{\alpha\beta}$ denotes the map

$$x_\alpha \circ x_\beta^{-1} : x_\beta\vdash(U_\alpha \cap U_\beta) \rightarrow x_\alpha\vdash(U_\alpha \cap U_\beta).$$

Such an atlas can be augmented by adjoining all charts (U,x) for M which are compatible with $\tilde{A}(M)$ in the sense that

$$\forall\alpha\varepsilon\tilde{I}, : U \cap U_\alpha \neq \emptyset \quad x \circ x_\alpha^{-1} : x_\alpha\vdash(U \cap U_\alpha) \rightarrow x\vdash(U \cap U_\alpha) \quad (3.1)$$

is C^∞.

The atlas $A(M)$ so obtained is said to be complete or maximal and provides M with a <u>differentiable structure</u>. A differentiable manifold is a topological manifold equipped with a differentiable structure defined by a complete atlas $A(M) = \{U_\alpha, x_\alpha\}_{\alpha\epsilon I}$. In general, a given topological manifold may admit more than one differentiable structure.

Given two manifolds M and N, a map $f: M \to N$ is differentiable at $p\epsilon M$, iff for any chart $(U,x)_p \epsilon A(M)$ and $(V,y)_{f(p)} \epsilon A(N)$ adapted to $p\epsilon U \subset M$ and $f(p)\epsilon V \subset N$, the map

$$y \circ f \circ x^{-1} : x_\vdash(U) \to y_\vdash(V)$$

is C^∞ at $x(p)$.

The compatibility condition (3.1) on the atlas structure ensures that the definition is independent of the choice of coordinate charts.

The map f is differentiable if and only if it is differentiable for all $p\epsilon M$, and f is a diffeomorphism if and only if f is bijective and $f^{-1}: N \to M$ is also differentiable. Moreover, a smooth map $f: M \to N$ is a local diffeomorphism at $p\epsilon M$, if and only if there is an open neighborhood U of $p\epsilon M$ such that the map $f|_U: U \to f(U)$ is a diffeomorphism.

An active transformation is a C^∞ mapping $f: M \to N$— which acts on and transforms the points of the manifold. The set of local diffeomorphisms of M into itself forms a pseudo-group of transformations. The composition of two local diffeomorphisms is only defined when the corresponding restricted diffeomorphisms have compatible domain and target. The passive version of the pseudo-group of local diffeomorphisms is the pseudo-group of coordinate transformations. A given neighborhood $\bar{U}$ of M is mapped by various coordinate maps onto different open neighborhoods of $\mathbb{R}^n$ and the coordinate transformations, which leave the points of M fixed, are local diffeomorphisms of $\mathbb{R}^n$, which can only be composed when the target of one is the domain of the other.

A manifold symmetry transformation is an active transformation $f: M \to M$, which is a diffeomorphism. The manifold symmetry group is the group of all diffeomorphisms of M.

An n-dimensional manifold whose only properties are those that fall under the concept of a manifold, Weyl interprets physically as an n-dimensional empty world.[25]

4.4 THE AFFINE STRUCTURE

Let TM_p denote the tangent space of M at $p\varepsilon M$. The transport of a vector $X_p \varepsilon TM_p$ to an infinitesimal neighboring point $p'\varepsilon M$ constitutes a <u>parallel displacement</u>, if and only if relative to some coordinate system $\bar{x}^i$, the resulting $X_{p'}\varepsilon TM_{p'}$ possesses the same components as the vector $X_p \varepsilon TM_p$. This definition of parallel displacement is coordinate dependent in that it characterizes parallel displacement as the type of transport of vectors which leaves their components unchanged relative to a specific coordinate system $\bar{x}^i$. For any arbitrary coordinate system x^i at $p\varepsilon M$ there corresponds, in the above fashion, a possible system of parallel displacements of a vector field at $p\varepsilon M$ to all infinitesimal neighboring points of $p\varepsilon M$. If we <u>arbitrarily</u> fix a definite coordinate system x^i at $p\varepsilon M$, then such a possible system of parallel displacement may be expressed as follows.

Let X_p^i denote the components of $X_p \varepsilon TM_p$ at p with coordinates x^i. If after parallel transport to some neighboring point p' with coordinates $x^i + dx^i$ the resulting vector $X_{p'}\varepsilon TM_{p'}$ has the components $X_p^i + dX_p^i$ then the changes of the components dX_p^i depend linearly on the vector X_p, that is,

$$dX_p^i = -d\Lambda_j^i X_p^j. \tag{4.1}$$

Moreover, the coefficients $d\Lambda_j^i$ depend linearly on the displacement vector or direction of the displacement dx^k, so that we have the linear form

$$d\Lambda^i_j = \Gamma^i_{jk} dx^k. \tag{4.2}$$

Hence the displaced vector $X_{p'} \varepsilon TM_{p'}$ whose components are the same as those of $X_p \varepsilon TM_p$ can be expressed as

$$X^i_{p'} = X^i_p + dX^i_p$$

or

$$dX^i_p = X^i_{p'} - X^i_p = -\Gamma^i_{jk} X^j_p dx^k. \tag{4.3}$$

Conversely, if for an arbitrary coordinate system x^i at $p \varepsilon M$

$$X^i_{p'} = X^i_p - \Gamma^i_{jk} X^j_p dx^k \tag{4.4}$$

then one can show easily that there exists a coordinate system $\bar{x}^i$ at $p \varepsilon M$, such that $\bar{X}^i_{p'} = \bar{X}^i_p$.

The magnitudes Γ^i_{jk} satisfy the symmetry condition

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{4.5}$$

and determine the whole process of parallel displacement. No further conditions are imposed upon them. If the

Γ^i_{jk} are some numbers satisfying (4.5), and if we define through (4.2) and (4.3) the transport of a vectorfield at p to all neighboring points of p', then this constitutes a possible system of infinitesimal parallel displacement; that is, there exists a coordinate system at $p\varepsilon M$ in which the components of the displaced vectors do not suffer any change under the defined transport.

A manifold M is endowed with an affine connection if the manifold structure is such that at each point $p\varepsilon M$ there exists one and only one concept of parallel displacement that may be characterized as the real one among the possible ones. The magnitudes Γ^i_{jk} which determine at every point $p\varepsilon M$ the actual or real infinitesimal parallel displacements of vectors, are called the components of the affine connection of M. At any arbitrary point $p\varepsilon M$ they can be made to vanish through a suitable choice of a coordinate system x^i for the neighborhood of $p\varepsilon M$. That is, we require of the concept of infinitesimal parallel displacement that for every $p\varepsilon M$, there exists a coordinate system x^i for the neighborhood of $p\varepsilon M$, such that the components of any vector at p are not altered by an infinitesimal parallel displacement. Weyl calls such a coordinate system at $p\varepsilon M$ a <u>geodetic coordinate system</u>.

This requirement characterizes the <u>nature</u> of an affine connection of M which is everywhere the same. That is, no difference exists among the possible points pεM with regard to the <u>nature</u> of the affine structure of M at those points and their neighborhoods. A manifold which is an affine manifold is homogeneous in this sense. Moreover, manifolds do not exist whose affine structure is of a different <u>nature</u>.

<u>An n-dimensional manifold M, which is an affinely connected manifold, we interpret physically as an n-dimensional world (spacetime) filled with a gravitational field.</u>[26]

4.5 THE PROJECTIVE STRUCTURE

The infinitesimal process of parallel displacements of vectors contains, as a special case, the infinitesimal displacement of a direction into its own direction. Such an infinitesimal autoparallelism of directions is characteristic of the projective structure of an affinely connected manifold. An infinitesimal autoparallelism of a direction R at an arbitrary point $p \varepsilon M$ consists in the parallel displacement of R at p to a neighboring point p' which lies in the direction R at p. A curve is geodesic if and only if its tangent direction R experiences infinitesimal autoparallelism when moved along all the points of the curve. The concept of a geodesic path constitutes an abstraction from affine geometry; it is definable exclusively in terms of autoparallelism of directions and not that of vectors. Roughly speaking, an affine geometry is essentially a projective geometry with the notion of distance defined along the curves. By eliminating all possible notions of distance along curves, or equivalently, all the parameter descriptions of the curves, one abstracts the projective geometry from affine geometry.

Consequently, one essential difference between a path and a curve is that, while both are one-dimensional submanifolds of M, a path does not depend in any way on the parameters used to describe it. A path is, therefore, sometimes defined as an equivalence class of curves under arbitrary parameter transformations. Hence, projective geometry may be defined as an equivalence class of affine geometries.

The set of curves which are solutions of a set of differential equations of the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{5.1}$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$ are functions only of $p \varepsilon M$, define the geometry of paths. Any curve $\gamma\colon I \to M (I \quad \mathbb{R})$, whose coordinates are given by

$$x^i \circ \gamma(s) \equiv \gamma^i(s)$$

and which satisfies (5.1), has the property that, for a sufficiently small neighborhood, each of the curve's points p is joined to any other of its points p' by one and only one curve, namely, that one which results from an autoparallelism of the direction R at p to the neighboring point p' that lies in the direction R at p.

The interest in projective geometry is, therefore, in the paths, that is, in the set of points of the curves which are solutions of (5.1), rather than in the possible parameter descriptions of curves. Hence, one may characterize projective geometry as the "true geometry of paths," since one is concerned with the properties of paths, apart from their characterization, by means of a particular affine connection and, hence, parameter description.

In physical spacetime the projective structure has an immediate intuitive significance according to Weyl. The real world is a non-empty world filled with an inertial-gravitational field, which Weyl calls the guiding field. It is an indubitable fact, according to Weyl,[27] that a body which is let free in a certain world direction (time-like 4-vector) carries out a uniquely determined natural motion from which it can only be diverted through an external force. The process of autoparallelism of directions appears, thus, as the tendency of persistence of the world direction of a free particle whose motion is governed by the guiding field. This natural motion occurs on the basis of an effective infinitesimal tendency of persistence, which parallelly displaces the world direction R of a body at an arbitrary point p on its trajectory to a neighboring point p' that lies in the direction R at p.

The projective structure of spacetime constitutes the inertial persistence of the guiding field which includes both the inertial and the gravitational field. If the motion of a particle is governed by a directing field, then such a particle's spacetime trajectory is determined uniquely by an event on the trajectory and its direction at that event. If such a field is a geodesic directing field, then such a particle's trajectory is geodesic. Thus, Weyl's guiding field is a geodesic directing field.[28]

If external forces exert themselves on a body, then a motion results which is determined through the conflict between the tendency of persistence due to the guiding field and the force. The tendency of persistence of the guiding field is a type of constraining guidance, which the inertial-gravitational field exerts on every body.

> Galilei's inertial law shows, that there exists a type of constraining guidance in the world which imposes on a body that is let free in some definite world direction a unique natural motion from which it can only be diverted through external forces; this occurs on the basis of an effective infinitesimal tendency of persistence from point to point, which parallelly transfers the world direction R of the body at an arbitrary point p to an infinitesimally close neighboring point p', that lies in the direction R at p.[29]

In Galilean spacetime, the guiding field constitutes a formal geometric structure that is rigidly tied to the spacetime manifold M. In the theory of Special Relativity, the guiding field is likewise a rigid formal geometric structure, appearing in the coordinate systems that are equivalent up to Lorentz-transformation. In both cases the guiding field acts on matter, but is not in turn affected by matter. In the theory of General Relativity the guiding field includes the gravitational field and, thus, ceases to be a rigid geometrical structural field. It is a flexible structural field that is physically real, analogous to the electromagnetic field and interacts dynamically with matter, not one-sidedly, but reciprocally. Einstein's recognition that gravity is not a force field existing in addition to a rigid inertia-determining world-geometry, but as part of the metrical and affine structure of spacetime, transformed the geometrical structural field from an absolute entity, rigidly tied to the spacetime manifold, into a dynamical structural field on M which reciprocally interacts with matter. Thus, as Weyl has emphasized, in the dualism of force and inertia, gravitation belongs on the side of inertia.[30]

4.6 PROJECTIVE AUTOMORPHISM

One may characterize the projective structure on a manifold M either in terms of an equivalence class of geodesic curves under arbitrary parameter diffeomorphisms, or one may take the process of autoparallelism of directions as fundamental in defining a projective structure. Weyl took the latter approach. In this section we shall present both points of view beginning with the first one.[31]

The geometry of paths, is the theory of a set of differential equations of the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (6.1)$$

where $\Gamma^i_{jk} = \Gamma^i_{kj}$ are functions only of $p \varepsilon M$. Any curve $\gamma\colon I \to M$, $(I \subset \mathbb{R})$, that satisfies (6.1) and whose coordinates are given by

$$x^i \circ \gamma(s) \equiv \gamma^i(s)$$

is called a geodesic curve. If the parameter is changed along each geodesic curve by a diffeomorphic transformation $s \to t(s)$, then (6.1) becomes

$$\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt}\right)\left(\frac{dt}{ds}\right)^2 + \frac{dx^i}{dt} \frac{d^2t}{dx^2} = 0 \qquad (6.2)$$

It is clear from (6.2) that the form of (6.1) is preserved if and only if the parameter transformation takes the form $s \to t(s) = as + b$, where a and b are constants. To find a set of equations of geodesic curves that are covariant under an arbitrary parameter transformation, we multiply (6.2) by dx^i/dt, interchange i and j and subtract, and get

$$\frac{dx^{[j}}{dt}\left(\frac{d^2x^{i]}}{dt^2} + \Gamma^{i]}_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right) = 0 \tag{6.3}$$

[the square brackets on the indices denote anti-symmetry or skew-symmetry, i.e. $A^{[i}B^{j]} = \frac{1}{2}(A^iB^j - B^iA^j)$. Round brackets on indices will denote symmetry, i.e. $A^{(i}B^{j)} = \frac{1}{2}(A^iB^j + A^jB^i)$.] Equation (6.3) can also be expressed as

$$\delta^{ij}_{kl}\left(\frac{d^2x^k}{dt^2} + \Gamma^k_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right)\frac{dx^l}{dt} = 0 \tag{6.4}$$

where

$$\delta^{ij}_{kl} \equiv \begin{vmatrix} \delta^i_k & \delta^j_k \\ \delta^i_l & \delta^j_l \end{vmatrix} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k .$$

A set of curves which satisfy (6.4) and hence (6.2) constitutes an equivalence class of curves under parameter transformations. Such an equivalence class represents a geodesic path.

We define a projective structure P on a manifold M to be an equivalence class of curves under arbitrary parameter transformations, namely, a set of curves that satisfy equations (6.3) or (6.4).

If two affine structures Γ and $\bar{\Gamma}$ yield the same projective structure of M under arbitrary parameter transformations, then

$$\delta^{ij}_{kl}\left(\frac{d^2x^k}{dt^2}+\Gamma^k_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right)\frac{dx^l}{dt}=\delta^{ij}_{kl}\left(\frac{d^2x^k}{dt^2}+\bar{\Gamma}^k_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right)\frac{dx^l}{dt}. \quad (6.5)$$

According to (6.5), Γ and $\bar{\Gamma}$ are projectively equivalent since their geodesics must coincide. A change from one affine structure Γ to another $\bar{\Gamma}$ that leads to the same projective structure is called a projective transformation. Having taken parameter independence as our starting point in defining a projective structure on M, we wish now to determine the transformation law of a projective transformation.

The form of (6.5) is unaffected by an arbitrary choice of parameter. We may therefore assume the same value of the parameter for both Γ and $\bar{\Gamma}$ at any point on the path. Equation (6.5) may therefore be written as

$$\delta^{ij}_{kl}\Gamma^k_{pq}dx^l dx^p dx^q = \delta^{ij}_{kl}\bar{\Gamma}^k_{pq}dx^l dx^p dx^q. \quad (6.6)$$

Symmetrizing in l, p and q we get

$$[(\delta^{ij}_{kl}\Gamma^{k}_{pq} + \delta^{ij}_{kp}\Gamma^{k}_{ql} + \delta^{ij}_{kq}\Gamma^{k}_{lp}) \tag{6.7}$$

$$- (\delta^{ij}_{kl}\overline{\Gamma}^{k}_{pq} + \delta^{ij}_{kp}\overline{\Gamma}^{k}_{ql} + \delta^{ij}_{kq}\overline{\Gamma}^{k}_{lp})]dx^{l}dx^{p}dx^{q} = 0$$

Since the choice of the direction at an arbitrary point of a path is arbitrary, (6.7) is an identity in the differentials, and we can write

$$\delta^{ij}_{kl}\Gamma^{k}_{pq} + \delta^{ij}_{kp}\Gamma^{k}_{ql} + \delta^{ij}_{kq}\Gamma^{k}_{lp} = \delta^{ij}_{kl}\overline{\Gamma}^{k}_{pq} + \delta^{ij}_{kp}\overline{\Gamma}^{k}_{ql} + \delta^{ij}_{kq}\overline{\Gamma}^{k}_{lp}. \tag{6.8}$$

Contraction on p and j yields

$$(n+1)\Gamma^{i}_{jk} - \delta^{i}_{j}\Gamma_{k} - \delta^{i}_{k}\Gamma_{j} = (n+1)\overline{\Gamma}^{i}_{jk} - \delta^{i}_{j}\overline{\Gamma}_{k} - \delta^{i}_{k}\overline{\Gamma}_{j} \tag{6.9}$$

where $\Gamma^{l}_{lk} \equiv \Gamma_{k}$.

Equation (6.1) is covariant under a coordinate transformation $x^{i} \to \overline{x}^{i}$, iff the transformation law for the Γ's is

$$\Gamma^{l}_{jk}\frac{\partial\overline{x}^{i}}{\partial x^{l}} = \overline{\Gamma}^{i}_{lp}\frac{\partial\overline{x}^{l}}{\partial x^{j}}\frac{\partial\overline{x}^{p}}{\partial x^{k}} + \frac{\partial^{2}\overline{x}^{i}}{\partial x^{j}\partial x^{k}}. \tag{6.10}$$

Contractions of (6.10) leads to

$$\Gamma_{j} = \overline{\Gamma}_{k}\frac{\partial\overline{x}^{k}}{\partial x^{j}} - \frac{\partial \ln\Delta(x,\overline{x})}{\partial x^{j}} \tag{6.11}$$

where $\Delta(x,x) \equiv |\frac{\partial x}{\partial x}|$.

Hence $(\bar{\Gamma} - \Gamma)_j = (\bar{\Gamma} - \Gamma)_k \frac{\partial \bar{x}^k}{\partial x^j}$. If we define

$$(\bar{\Gamma} - \Gamma)_j \equiv (n+1)\phi_j \tag{6.12}$$

where ϕ_j is some covariant vector field, we may reexpress (6.9) as

$$(\bar{\Gamma} - \Gamma)^i_{jk} - \delta^i_j\phi_k - \delta^i_k\phi_j = 0. \tag{6.13}$$

We have thus obtained the projective transformation law from one affine connection to another (in the same projective structure), namely

$$\Gamma^i_{jk} \overset{P}{\to} \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\phi_k + \delta^i_k\phi_j. \tag{6.14}$$

To study a projective structure, one can either work with a representative connection Γ and then check the invariance of all relevant properties under an arbitrary projective transformation (6.14) or one may introduce the projective coefficients

$$\Pi^i_{jk} \equiv \Gamma^i_{jk} - \frac{2}{n+1}\,\delta^i_{(j}\Gamma_{k)}. \tag{6.15}$$

To obtain (6.15) one need only choose ϕ_k in (6.12) such that $\bar{\Gamma}_k = 0$ and $-\Gamma_k = (n+1)\phi_k$ and use (6.9). The projective coefficients are invariant under a projective transformation and have zero trace ($\Pi^i_{ji} = 0$) but they do not constitute a connection.

A projective structure on a manifold may thus also be defined to be an equivalence class of projectively related affine connections. The projective coefficients denote that equivalence class, that is

$$\Pi^{i}_{jk}(\Gamma) = \Pi^{i}_{jk}(\overline{\Gamma})$$

for all Γ^{i}_{jk} and $\overline{\Gamma}^{i}_{jk}$, such that

$$\Gamma^{i}_{jk} \overset{P}{\rightarrow} \overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \delta^{i}_{j}\phi_{k} + \delta^{i}_{k}\phi_{j}.$$

Weyl took the process of autoparallelism of direction as fundamental in defining a projective structure of an affinely connected manifold.[32] The following is a necessary and sufficient condition that the projective structure is preserved under a transformation $\Gamma \rightarrow \overline{\Gamma}$.

C: A transformation $\Gamma \rightarrow \overline{\Gamma}$ preserves the projective structure P of an affine manifold M if and only if $(\overline{\Gamma} - \Gamma)^{i}_{jk}X^{j}X^{k} \propto X^{i}$, where X^{i} is an arbitrary vector.

Condition C says that a change in the affine structure of M preserves the projective structure of M, if and only if the resulting vectors $X^{i}_{p}, \varepsilon TM_{p}$, and $\overline{X}^{i}_{p}, \varepsilon TM_{p}$, of the respective autoparallelism under Γ and $\overline{\Gamma}$, differ at most in length but not in direction. That is, the respective autoparallelisms are identical.

Weyl proved the following theorem: A transformation $\Gamma \to \overline{\Gamma}$ satisfies $(\overline{\Gamma} - \Gamma)^i_{jk}X^jX^k \propto X^i$ if and only if $(\overline{\Gamma} - \Gamma)^i_{jk} = \delta^i_j\phi_k + \delta^i_k\phi_j$, where ϕ_k is some covariant vector field.

Assume $(\overline{\Gamma} - \Gamma)^i_{jk} = \delta^i_j\phi_k + \delta^i_k\phi_j$. Then

$$(\overline{\Gamma} - \Gamma)^i_{jk}X^jX^k = (\delta^i_j\phi_k + \delta^i_k\phi_j)X^jX^k$$

$$= (\phi_j X^j)X^i.$$

To show the converse we express $(\overline{\Gamma} - \Gamma)^i_{jk}X^jX^k \propto X^i$ as

$$X^{[p}(\overline{\Gamma} - \Gamma)^{i]}_{jk}X^jX^k = 0.$$

or as

$$\delta^{[p}_q(\overline{\Gamma} - \Gamma)^{i]}_{jk}X^jX^kX^q = 0. \tag{6.16}$$

Symmetrizing in j, k, and q and then contracting over p and q gives

$$\delta^{[p}_q(\overline{\Gamma} - \Gamma)^{i]}_{jk} + \delta^{[p}_j(\overline{\Gamma} - \Gamma)^{i]}_{pk} + \delta^{[p}_k(\overline{\Gamma} - \Gamma)^{i]}_{jp} = 0$$

or

$$(n+1)(\overline{\Gamma} - \Gamma)^i_{jk} - \delta^i_j(\overline{\Gamma} - \Gamma)_k - \delta^i_k(\overline{\Gamma} - \Gamma)_j = 0 \tag{6.17}$$

Defining

$$(\bar{\Gamma} - \Gamma)_k \equiv (n+1)\phi_k \qquad (6.18)$$

and substituting (6.18) into (6.17) gives

$$(\bar{\Gamma} - \Gamma)^i_{jk} = \delta^i_j\phi_k + \delta^i_k\phi_j. \qquad (6.19)$$

As before, we choose ϕ_k such that $\bar{\Gamma}_k = 0$ and $-\Gamma_k = (n+1)\phi_k$. Substituting the latter relation into (6.19) we get

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{2}{(n+1)}\,\delta^i_{(j}\Gamma_{k)}, \qquad (6.20)$$

where

$$\Pi^i_{ik} = \bar{\Gamma}_k = 0; \quad \Pi_{i\,[jk]} = 0. \qquad (6.21)$$

Again, Π^i_{jk} is a representative of the equivalence class of projectively equivalent connections.

Consider now the standard equation for the geodesics of a symmetric linear connection

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

Using (6.20) we have

$$\frac{d^2x^i}{dt^2} + \Pi^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = \left(- \frac{2}{(n+1)} \Gamma_j \frac{dx^j}{dt}\right)\frac{dx^i}{dt}$$

$$\frac{d^2x^i}{dt^2} + \Pi^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = \mu(t)\frac{dx^i}{dt} \tag{6.22}$$

$$\frac{dx^{[1}}{dt}\left(\frac{d^2x^{i]}}{dx^2} + \Pi^{i]}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt}\right) = 0.$$

Equation (6.22), with $\mu(t)$ denoting an arbitrary but not necessarily affine parameter transformation, has the same form as the standard equation for geodesics of a symmetric linear connection. But the projective coefficients do not form the components of a linear connection; they transform differently due to (6.21). Therefore the paths described by (6.22) do not admit a preferred parametrization. The condition $\mu(t) = 0$ is not covariant under coordinate transformation. Rather there exists locally an equivalence class of projectively equivalent symmetric linear connections with correspondingly different parameter assignments. That class is denoted by Π^i_{jk} in (6.20).

4.7 THE METRIC STRUCTURE

A good way to appreciate Weyl's view of the metric structure of spacetime is to examine some aspects of his ingenious attempt to establish a unified field theory of gravitation and electromagnetism in terms of a gauge-invariant geometry. Although Weyl had to abandon his specific efforts in this direction with the advent of quantum theory, Weyl's new model of differential geometry (he called it pure infinitesimal geometry) has great theoretical and aesthetic appeal, and was soon applied in other domains of physics, especially in quantum theory, where it proved to be of real importance.

One of the distinguishing features of long-range forces is that their magnitudinal increases or decreases is inversely proportional to the square of the distance between the interacting bodies. Short-range forces on the other hand, increase or decrease exponentially. Gravitational and electromagnetic forces are both long-range forces.

Einstein geometrized the gravitational field by connecting it with the Riemannian metric structure. By replacing the potential of the gravitational force with the metric potentials g_{ij}, the components of the metric tensor, Einstein transformed the gravitational field into the basic geometric structure of spacetime. In the general theory of relativity the gravitational field is thus accounted for in terms of the curvature of spacetime, but the electromagnetic field remains completely unrelated to the spacetime geometry. It was therefore natural to suggest that the electromagnetic field might also be ascribed to some property of spacetime, instead of being merely something embedded in spacetime. Since, however, the components g_{ij} of the metric tensor are already sufficiently determined by the field equations, this would require setting up a more general differential geometry than the one which underlies Einstein's theory, in order to make room for incorporating electromagnetism into spacetime geometry. Such a generalized differential geometry would describe both long-range forces, and a new theory based on this geometry would constitute a unified field theory of electromagnetism and gravitation.[33]

In 1918, Weyl proposed such a theory by generalizing Riemannian differential geometry in a way that allowed for a greater flexibility in choosing the metric tensor. By disallowing an absolute comparison of length at two different points that are separated by a finite amount, Weyl achieved a flexibility which proved to be just enough for the integration of the entire formalism of the theory of electromagnetism into the spacetime geometry. In such a geometry, comparisons of length can only be made with respect to a specified path joining two points not infinitesimally close together, and parallel displacement along different paths will generally lead to different results for the ratio of two elements of length.

That is, length transfer is non-integrable.

In a Riemannian space the concept of parallel displacement is defined by two conditions:

(1) The components of a vector remain unchanged for an infinitesimal parallel displacement in a suitably chosen coordinate system.

(2) The length of a vector X_p at $p \varepsilon M$ remains unchanged during an infinitesimal parallel displacement to points p' infinitesimally close to p.

The first condition is satisfied if

$$dX_p^i = -\Gamma^i_{jk} X_p^j dx^k \tag{7.1}$$

and $\Gamma^i_{jk} = \Gamma^i_{kj}$. At each point there exists a suitable coordinate system in which the affine components vanish. In general the result of parallel displacement is path dependent, that is, non-integrable. A vector X_p at $p \varepsilon M$ which is carried around a closed circuit by a continual parallel displacement back to p, will not in general return to its initial direction at p. For a closed loop which circumscribes an infinitesimally small portion of space, the rotation of the vector per unit area constitutes the measure of the local curvature of space. Whether or not an affinity on M is integrable, that is, path independent, depends on whether or not the curvature tensor vanishes.

The second condition says that a Riemannian space possesses a definite affine connection which is uniquely determined by the metric. Among the possible systems of parallel displacements of a vector X_p at $p \varepsilon M$ to infinitesimally near points p', there exists one, and only one system of parallel displacement which is length preserving. This means that

$$d(g_{ij}X^i X^j) = 0. \tag{7.2}$$

Consequently,

$$\Gamma_{ijk} + \Gamma_{jik} = g_{ij,k} \tag{7.3}$$

where the

$$\Gamma_{ijk} = \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) \tag{7.4}$$

denote the Christoffel symbols of the first kind.

In Riemannian geometry then, finite parallel displacement of direction is non-integrable and, as the above considerations show, the metric uniquely determines the affinity of the manifold. Now Riemann and Einstein make the additional assumption that length transfer is integrable. Consequently, two vectors that are separated by a finite amount may be compared with respect to length. It is this assumption that Weyl rejects and considers as entirely unwarranted from a mathematical point of view.

Weyl says:

> In a geometry, which is pure infinitesimal geometry, such an assumption is clearly inadmissable. It must be replaced by the more general one, viz., that the length can only be congruently transferred infinitesimally. We have to take into account that congruent transfer of length along a path which connects the points that are separated by a finite amount may prove to be path dependent.[34]

If a manifold M is such that line elements or vectors at some point $p \epsilon M$ may be compared with respect to their length at p, then M possesses a distance measure or metric at p. Thus every vector $X_p \epsilon TM_p$ defines an interval or distance at p and there exists a non-degenerate quadratic form Q at p such that X_p, $Y_p \epsilon TM_p$ define congruent intervals if and only if $Q(X_p, Y_p)$, that is, if and only if $\|X_p\|^2 = \|Y_p\|^2$. This non-degenerate quadratic form is determined only up to a non-zero and positive (or negative) proportionality factor $\lambda(x^i)$. If M has w positive and z negative dimensions such that $w + z = n$, then λ must be positive if $w \neq z$, in order that the topological structure (signature) of M be preserved. Since M will represent physical spacetime, ~~we will~~ forthwith assume $\lambda(x^i)$ to be a positive non-zero function of position.

To fix a value for λ at p is to calibrate M at p. λ is also called a gauge. A definite value of λ at p determines a standard of length at p and $\|X_p\|^2$ will denote the numerical value 1 of the distance measure of the interval defined by X_p. A change of the gauge factor at p will result in a new value of length according to

$$\bar{l} = \lambda l; \quad \lambda \neq 0. \tag{7.5}$$

In order that a manifold is a metric manifold, it is not sufficient that it possesses at each of its points a distance measure g_{ij} which determines the length of vectors or line elements at those points in the way just described. To count as a metric space, a manifold must in addition satisfy the condition of metrical <u>connectedness</u>. If M is a metric manifold, then each of its points is metrically connected with the points in its immediate neighborhood. <u>A point $p \varepsilon M$ is metrically connected with its immediate neighborhood, if and only if for every interval at p, an interval at p' is determined to which an interval at p gives rise when it is congruently displaced from p to p', where p' is infinitesimally close to p.</u> The only condition imposed on the concept of metric connectedness at $p \varepsilon M$ is the following: the neighborhood U_p of $p \varepsilon M$ can always be calibrated in such a way that the length of every interval at $p \varepsilon M$ does not suffer any change when congruently transplanted to some infinitesimally close point $p' \varepsilon U_p \subset M$. Such a calibration or gauge Weyl calls geodetic at $p \varepsilon M$.

Note that the definition of a metrical connection is similar to the definition of an affine connection. The latter says: A point $p \varepsilon M$ is affinely connected with its immediate neighborhood, if and only if for every vector X_p at $p\varepsilon M$, a vector $X_{p'}$ at p' is determined to which a vector X_p gives rise when it is displaced in parallel from p to p', where p' is infinitesimally close to p.

If the manifold is calibrated everywhere, then $\lambda(x^i)$ is a scalar field on M. For an arbitrary calibration of M, if l is the value of the distance measure of an arbitrary interval at p and if l+dl is the value of the distance measure of the interval congruently displaced to an infinitely near point p', then the relation

$$dl = -l d\theta \tag{7.6}$$

holds necessarily, where $d\theta$ is independent of the displaced interval. If the gauge at $p\varepsilon M$ and the neighborhood is changed according to

$$\bar{l} = \lambda l \tag{7.7}$$

then

$$d\bar{l} = -\bar{l} d\bar{\theta} \tag{7.8}$$

and

$$d\bar{\theta} = d\theta - \frac{d\lambda}{\lambda} \tag{7.9}$$

$d\bar{\theta}$ vanishes identically at p for some value of λ and some displacement dx^i, if and only if $d\theta$ is of the differential form

$$d\theta = \theta_i dx^i. \tag{7.10}$$

The similarities between the process of parallel displacement of vectors and congruent transfer of intervals can be seen immediately by comparing $dl = -\theta_i l dx^i$ with $dX^i = -\Gamma^i_{jk} X^j dx^k$.

In Weyl's geometry there appears, besides the well known quadratic differential form $ds^2 = g_{ij} dx^i dx^j$ which defines the metric at each individual point, another linear form, $\theta_i dx^i$, which determines the metric relationship between a point and its neighborhood. Weyl says:

> The metrical character of a manifold is therefore characterized relatively to a system of reference (=coordinate system and calibration) by two fundamental terms, namely, a quadratic differential form $Q = \sum_{ik} g_{ik} dx_i dx_k$ and a linear one, $d\phi = \Sigma \phi_i dx_i$.[35]

Both differential forms must not only be invariant under arbitrary coordinate transformations but must also be gauge invariant. That is, if the gauge is changed then we also have to postulate their gauge invariance with respect to the substitutions

$$\bar{g}_{ij} = \lambda g_{ij}; \quad \bar{\theta}_i = \theta_i - \frac{1}{\lambda}\frac{\partial \lambda}{\partial x^i}. \tag{7.11}$$

The fundamental principle of infinitesimal geometry says that the metric structure on a manifold M uniquely determines the affine structure on M. Weyl shows that this fundamental principle not only holds for a Riemannian metric space, but is also valid for a metric space of a more general nature. That is, "the principle of transference of distances or length which is the basis of metrical geometry, carries with it a principle of transference of direction; <u>in other words, an affine relationship is inherent in metrical space</u>."[36]

To show this, consider again the parallel displacement law

$$dX^i_p = -\Gamma^i_{jk}X^j dx^k. \qquad (7.12)$$

Since $l = g_{ij}X^iX^j$ and $dl = -l d\theta$, we have

$$\begin{aligned} -g_{ij}X^iX^j d\theta &= d(g_{ij}X^iX^j) \\ &= dg_{ij}X^iX^j + g_{ij}dX^iX^j + g_{ij}X^i dX^j. \qquad (7.13) \end{aligned}$$

Making use of (7.12) we have

$$g_{ij}\theta_k X^iX^j dx^k + g_{ij,k}X^iX^j dx^k = \Gamma_{ijk}X^jX^i dx^k + \Gamma_{jik}X^jX^i dx^k$$

where $\Gamma_{ijk} = g_{li}\Gamma^l_{jk}$. Since the above identity must hold for arbitrary choices of X^i and dx^j

$$\Gamma_{ikj} + \Gamma_{jki} = g_{jk,i} + g_{jk}\theta_i . \qquad (7.14)$$

Performing cyclical permutations on (7.14) twice then adding the results and subtractions (7.14) from the sum yields

$$\Gamma_{kij} = \tfrac{1}{2}(g_{ki,j} + g_{jk,i} - g_{ij,k}) + \tfrac{1}{2}(g_{ki}\theta_j + g_{jk}\theta_i - g_{ij}\theta_k).$$

Hence

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + \tfrac{1}{2}(\delta^k_i\theta_j + \delta^k_j\theta_i - g_{ij}\theta^k) \qquad (7.15)$$

where $\left\{ {k \atop ij} \right\}$ is the Christoffel symbol of the second kind. Therefore the components of the affine connection are uniquely determined by the metric connection of the gauge-invariant Weyl geometry.

We shall now briefly indicate a possible physical interpretation of this geometry within the context of Weyl's efforts to establish a unified field theory of gravitation and electromagnetism.

If a vector of unit length is congruently displaced along some curve (connecting points p and q, then its final unit of length can be computed by integrating along C_{pq} and

$$l_q = l_p e^{-\int_{C_{pq}} d\theta} \qquad (7.16)$$

where l_q and l_p are the units of length at p and q respectively, and the exponential represents the real factor associated with the path C_{pq}.

If $d\theta$ is a <u>total differential</u>, then the length of a vector (and hence the value of the integral) becomes independent of the path along which it is transfered (integrated). This is of course what happens in an ordinary Riemannian metric space. A necessary and sufficient condition that $d\theta$ is a total differential is that

$$F_{ij} \equiv \theta_{j,i} - \theta_{i,j} = 0. \tag{7.17}$$

It follows from the above that

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0 \tag{7.18}$$

for $i \neq k \neq j$; $i,j,k = 0,1,2,3$.

Now the condition that θ_i is a gradient vector field, (i.e. $d\theta = \theta_i dx^i$ is a total differential) is of the same form as the equations for the four-potential

$$\underset{\sim}{V} = (V, -\frac{1}{c}\underset{\sim}{A})$$

of the electromagnetic field. Consequently, the identification of the gradient vector field θ_i with the electromagnetic potential $\underset{\sim}{V}$ (the condition that $d\theta$ be a total differential) is equivalent to the requirement that the electromagnetic field $\underset{\sim}{E}$, $\underset{\sim}{H}$ be zero, since $H, E = 0$ if and only if the linear differential form $d\underset{\sim}{V} = -cVdt + A_i dx^i$ $(i=1,2,3)$ is a total differential. Thus if $d\underset{\sim}{V}$ is not a total differential, or equivalently, if $H, E \neq 0$, then any transfer of clocks or rods around a closed path is path dependent, that is, $\oint A_i dx^i \neq 0$. If physical spacetime corresponds to a Weyl geometry, then two synchronized and identical clocks at an event p which are separated and moved along different world lines to an event q, will not only differ with respect to the elapsed time (first clock effect), but in general the two clocks will tick at different rates at q (second clock effect). That is, time transport in a Weyl space is such that two congruent time intervals at p will not in general be congruent at q, when displaced in parallel from p to q along different paths.

As Einstein pointed out, however, it is precisely this situation which suggests that Weyl's geometry conflicts with experience. In Weyl's geometry the frequency of the spectral lines of atomic clocks would depend on the location and past histories of the atoms. But experience teaches otherwise. The spectral lines are well defined and sharp. Atomic clocks define units of time and experience shows that they are integrably transported. Thus if we assume that the atomic and gravitational standard time as defined in a Weyl geometry (affine-parameter of Γ^{i}_{jk}) are identical, then $F_{ij} = 0$. But if $F_{ij} = 0$ then a Weyl geometry reduces to the standard Riemannian geometry that underlies General Relativity. The vanishing of Weyl's <u>Streckenkrümmung</u> $\underset{\sim}{F}$ is necessary and sufficient for the existence of a Riemannian metric g_{ij}, such that Γ is metric with respect to g_{ij} and g_{ij} is compatible with the conformal (causal) structure.

4.8 THE CONFORMAL STRUCTURE

We saw in Section (4.5) that Weyl took the affine structure as basic and considered the projective structure as arising from it by abstraction. In an analogous manner he considers the conformal structure as arising from the metric structure by abstraction.

The defining characteristic of a conformal structure is given by the equation

$$g_{ij}dx^i dx^j = 0. \tag{8.1}$$

Let X, $Y \varepsilon TM_p$. Then the angle between the two vectors is measured by the ratio

$$\frac{X^i Y_i}{\|X\|\,\|Y\|} = \frac{g_{ij}X^i Y^j}{[(g_{ij}X^i X^j)(g_{ij}Y^i Y^j)]^{\frac{1}{2}}}.$$

Clearly this ratio does not change under the gauge transformation $\bar{g}_{ij} = \lambda g_{ij}$. The gauge transformation is angle preserving, that is conformal. In Weyl's gauge-invariant geometry the metric tensor g determines angles only and one needs in addition the specification of the gauge vector θ_i in order to be able to measure length. The conformal structure of M determines the bilinear quadratic form whereas the linear form $\theta_i dx^i$ remains free. Therefore, every virtual change of the metric field of M which preserves the conformal structure, leaves the bilinear quadratic form unchanged and is consequently only effected by some variation of the linear form $\theta_i dx^i$. The change of the components of the affine connection under a change of the metric structure of M which preserves the conformal structure is easily shown to be given by

$$
\begin{aligned}
(\overline{\Gamma} - \Gamma)^{i}_{jk} &= \tfrac{1}{2}(\delta^{i}_{j}\theta_{k} + \delta^{i}_{k}\theta_{j} - g_{jk}g^{il}\theta_{l}) \\
&= \tfrac{1}{2}(\delta^{i}_{j}\theta_{k} + \delta^{i}_{k}\theta_{j} - g_{jk}\theta^{k})
\end{aligned}
\qquad (8.2)
$$

<u>A 4-dimensional manifold M endowed with a conformal structure of Lorentzian type we interpret physically as a non-empty 4-dimentional world filled with a "causal field."</u>

4.9 WEYL'S GROUP-THEORETICAL ACCOUNT OF METRIC STRUCTURES AND HIS INTERPRETATION OF RIEMANN'S DYNAMICAL HYPOTHESIS

This Section provides a brief outline of Weyl's group theoretical characterization of metric structures in his effort to prove that among all the possible metrics that can be put on a continuous manifold, the Pythagorean-Riemannian metric is, in important respects, unique.[37] The reason for this aside is three-fold: It will clarify some of the conceptual issues Riemann was groping with in his attempt to justify the Pythagorean nature of the metric, and will bring into sharp focus Weyl's understanding of Riemann's Dynamical Hypothesis. This will in turn make explicit to what extent Weyl's realism is a natural outgrowth of Riemann's work and that his interpretation of Riemann's inaugural lecture could not possibly be in accord with that of Grünbaum.

We saw in Section 7, that in a Riemannian space the concept of parallel displacement is defined by two conditions:

(1) The components of a vector remain unchanged for an infinitesimal displacement in a suitably chosen coordinate system.

(2) The length of a vector X_p at $p\varepsilon M$ remains unchanged during an infinitesimal parallel displacement to points p' infinitesimally close to $p\varepsilon M$. The first condition is satisfied if

$$\text{(i)} \quad dX_p^i = -\Gamma^i_{jk} X_p^j dx^k$$

(ii) $\Gamma^i_{jk} = \Gamma^i_{kj}$

It follows from the above conditions ((1) and (2)), that a Riemannian space possesses a definite affine connection which is uniquely determined by the metric. That is, among the possible systems of parallel displacements of a vector X_p at $p \varepsilon M$ to infinitely near points $p' \varepsilon M$ (i.e. for possible Γ^i_{jk}'s), there exists one and only one system of parallel displacement which is length preserving.

Weyl conjectured that the converse is also true: If there exists on a manifold such a relation between congruent transport and parallel displacement such that the metric uniquely determines the affine connection (i.e., if for some metric on M there exists among all possible systems of parallel displacements one and only one which is at the same time a congruent displacement), then such a metric is necessarily the Pythagorean-Riemannian metric.

The defining property of the metric consists in the concept of congruence. Within the context of Riemann's infinitesimal geometric standpoint, the concept of congruence must now be understood infinitesimally. If a manifold M is such that infinitesimal line elements or vectors at some point $p \varepsilon M$ may be compared with respect to their length at $p \varepsilon M$, then M possesses a distance measure or metric at $p \varepsilon M$. More generally we can say, that the metric is known at some point $p \varepsilon M$ when it is known which among the linear maps which map the tangent space at $p \varepsilon M$

onto itself, are the congruent linear maps.

In order that a manifold is a metric manifold, it is not sufficient that it possesses at each of its points a distance measure which determines the length of vectors or line elements at those points in the way just described. As we pointed out in Section 7, to count as a metric space, a manifold must in addition satisfy the condition of metrical connectedness. That is, if M is a metric manifold, then each of its points is metrically connected with the points in its immediate neighborhood. We say that a point $p\varepsilon M$ is metrically connected with its immediate neighborhood, if and only if for every interval at p, an interval at p' is determined to which an interval at p gives rise when it is congruently displaced from p to p', where $p'\varepsilon M$ is infinitesimally close to $p\varepsilon M$. The only condition imposed on the concept of metric connectedness of $p\varepsilon M$ is the following: the infinitesimal neighborhood U_p of $p\varepsilon M$ can always be calibrated in such a way that the length of every interval at $p\varepsilon M$ does not suffer any change when congruently transplanted to some infinitesimally close point $p'\varepsilon U_p \subset M$. That is, the metrical connection of a manifold is of the same type everywhere. The manifold is homogeneous to that extent. Put differently, at every point $p\varepsilon M$ there exists among the possible coordinate systems a geodetic coordinate system, such that, with respect to it, infinitesimal transport of all vectors at $p\varepsilon M$ to a neighboring point $p'\varepsilon M$ leaves their components unchanged and constitutes a

<u>congruent</u> displacement.

It is clear that for a Riemannian structure the congruent linear maps of the tangent space at $p_o \epsilon M$ onto itself forms a group which is isomorphic to the orthogonal group O(n). The metric at $p_o \epsilon M$ is therefore determined through the concrete realization of the orthogonal group at $p_o \epsilon M$.

There exists as many different types or forms of metrics or measure determinations at p_o as there exist <u>essentially different</u> groups of linear transformations. (Groups which are <u>essentially different</u> do not merely differ in terms of their coordinate representation.) The group of linear transformations which determines the Pythagorean-Riemannian metric, is the orthogonal group which leaves the fundamental quadratic differential form invariant.

Weyl extends the group-theoretical characterization of the metric (of whatever <u>form</u> or <u>type</u>) at some point $p_o \epsilon M$ to the notion of <u>metric connectedness</u> of M, that is, to the congruent displacement of a tangent space centered at $p_o \epsilon M$ to

(a) some point $p \epsilon M$ infinitely near to $p_o \epsilon M$

(b) arbitrary points $p \epsilon M$ infinitely near to $p_o \epsilon M$

and makes the following observations with respect to (a) and (b) respectively:

(a) All congruent displacements A from $p_o \epsilon M$ to $p \epsilon M$ consist of the composition of <u>one</u> congruent displacement A_o and an arbitrary rotation (microsymmetry, (see Section A8)) G_o at $p_o \varepsilon M$ which preceeds A_o. That is $A = A_o \circ G_o$, where $G_o \varepsilon \mathcal{G}_o$, and $\mathcal{G}_o$ is the microsymmetry group at $p_o \varepsilon M$.

If we consider a set of vectors in a tangent space at $p_o \varepsilon M$ occupying two positions that are congruent to one another, then the same congruent displacement A_o will map the latter into two positions at $p \varepsilon M$ that are congruent to one another. The microsymmetry group $\mathcal{G}$ at $p \varepsilon M$ is therefore equal to $A_o \circ \mathcal{G}_o \circ A_o^{-1}$. From the notion of metric connectedness one obtains that the microsymmetry group at some point $p \varepsilon M$ differs from $p_o \varepsilon M$ only with respect to orientation. Since one can continuously go from point $p_o \varepsilon M$ to some other point of the manifold, the microsymmetry groups on M are all of the same type i.e. they are all isomorphic to the abstract microsymmetry group. The manifold is therefore homogeneous in this respect:

At each of its points M admits a "geodetic" coordinate system such that the infinitesimal transport of all vectors at some point $p_o \varepsilon M$ to some infinitely near point $p \varepsilon M$ leaves their components unchanged and constitutes a congruent displacement. Put differently, at each point $p_o \varepsilon M$ a "geodetic" coordinate system may be introduced such that for some infinitesimal neighboring point $p \varepsilon M$ the microsymmetry group $\mathcal{G}$ at $p \varepsilon M$, has the same orientation, as the microsymmetry group $\mathcal{G}_o$ at $p_o \varepsilon M$.

(b) If one considers two successive infinitesimal congruent displacements of tangent vectors from p_o with coordinates x_o^i to the point p with coordinates x^i and a subsequent congruent displacement through δx^i, then the resultant congruent infinitesimal transport arises from the resultant displacement $dx^i + \delta x^i$.

A congruent transport is infinitesimal, if the change $d\xi^i$ of the components ξ^i of an arbitrary vector are infinitesimal and of the same order as the components dx^i of the displacement of the centre at p_o. Hence

$$d\xi^i = \varepsilon \cdot \sum_k \Lambda^i_{k1} \xi^k$$

constitutes an arbitrary infinitesimal congruent transport in the direction of the first coordinate axis to the points $(x_o^1 + \varepsilon, x_o^2, \ldots, x_o^n)$, and the coefficients $\Lambda^i_{k2}, \ldots, \Lambda^i_{kn}$ have the corresponding meaning for the infinitesimal congruent transport in the direction of the second to the nth coordinate axis. ε is an infinitesimal constant. Hence

$$d\xi^i = \sum_{k,r} \Lambda^i_{kr} \xi^k (dx)^r$$

constitutes the expression for a system of infinitesimal congruent transport to all points in the neighborhood of p_o. (Λ^i_{kr} denotes n^3 arbitrary coefficients).

To summarize what we have so far: The congruent linear maps of a tangent space at some point of the manifold form a group, namely the microsymmetry group. The nature of the metric at each point of the manifold is determined by a specific group of linear transformations, the microsymmetry group, which is a subgroup of GL(n). In a metric space, the microsymmetry groups at each point, which determine the same nature of the metric at each point, differ from one another only with respect to their orientation. They are all of the same type, in the sense that they determine the nature of the metric which is everywhere the same. The manifold is homogeneous in this respect.

However, the nature of the metric, which is everywhere the same on the manifold, does not determine the metrical connection from point to point and therefore does not determine the mutual orientation of the micro-symmetry groups at different points on the manifold.

By applying this formal group-theoretical characterization of metric structures to physical space Weyl is able to re-express Riemann's Dynamical Hypothesis (that binding forces may determine the metric structure of space) in the following way: The metrical connection from point to point of space (and hence the possible mutual orientations of the microsymmetry groups at different points in space) is determined not through the nature of the metric (which is everywhere the same), but the mutual orientations of the microsymmetry groups at different points are causally dependent on the material content of space, being free and consequently subject to arbitrary virtual change.

Weyl calls the preceding statement the Postulate of Freedom. It may be expressed more formally as follows.

Postulate of Freedom: Let M be endowed with a metric structure. Then the nature of the metric at each point $p \varepsilon M$ admits every possible metrical connection in the sense that for a given microsymmetry group at $p_o \varepsilon M$, which characterizes the nature of the metric of M, a metrical connection of $p_o \varepsilon M$ with points p in its infinitesimal neighborhood is possible, such that

$$d\xi^i = \sum_{k,r} \Lambda^i_{kr} \xi^k (dx)^r$$

constitutes a system of congruent transport to infinitesimally neighboring points for n^3 arbitrary given numbers Λ^i_{kr}.

The Postulate of Freedom as stated above succinctly expresses Riemann's dynamical view of geometry. It says that the metrical connection from point to point (hence the mutual orientations of the microsymmetry groups) is causally dependent on the material content filling space. That is, the coefficients Λ^i_{kr} in the above expression are determined by the distribution of matter. They are not rigidly tied to each point of the manifold but continuously vary from point to point.

Whatever the nature of the metric may be, that is, whatever the specific type of microsymmetry group that characterizes the nature of the metric of a metric space (the nature is everywhere the same since the very notion of measurability requires that the metric manifold be homogeneous at least to that extent), the metric connection (though of the same type everywhere) is quantitatively expressed through the coefficients Λ^{i}_{kr} and varies (inhomogeneously) from point to point due to the (inhomogeneous) distribution of matter in space.

What has been said until now, constitutes merely a conceptual analysis, that is, an explication of what lies in the concepts metric, metric connection, and parallel displacement.

What is not and what cannot be achieved on the basis of mere conceptual explication of the above concepts is to prove that among the various kinds of possible metrical structures that can be put on a continuous manifold representing physical space, the Pythagorean-Riemannian metric is unique.

Some claim which goes beyond mere concept analysis has to be made in order to provide some justification of one type of metric from alternative ones.

Weyl suggests that what renders the status of the Pythagorean-Riemannian metric unique concerns an additional postulate that goes beyond mere concept analysis. This postulate says: Whatever quantitative determination the essentially free metric connection may realize according to the Postulate of Freedom, for any given realization, that is, for any possible definite quantitative value, there exists among the possible systems of parallel displacements one and only one, which is at the same time a system of congruent transport. Put differently the postulate says: For a given metric form, the set of all congruent transports at a point $p \varepsilon M$ contains one and only one which is also affine. That is, given the set of coefficients $\{\Lambda^i_{jk}\}$, then for each Λ^i_{jk}, there is a unique A^i_{jk}, such that

$$\Gamma^i_{jk} = \Lambda^i_{jk} + A^i_{jk} = \Gamma^i_{kj},$$

where Γ^i_{jk} is the same for all Λ^i_{jk}.

This postulate is identical in content with what was earlier characterized as the fundamental theorem of infinitesimal geometry which says:

> the metric connection uniquely determines
> the affine connection.

We saw in Section 7 that in a Riemannian space this postulate is satisfied: If M is endowed with a Pythagorean-Riemannian metric, then there exists among the possible systems of parallel displacements one and only one, which is at the same time a system of congruent transport. What needs to be shown in order to establish the uniqueness of the Pythagorean-Riemannian metric is the converse: If a manifold is such that among all possible systems of parallel displacements there exists one and only one which is at the same time a system of congruent transport then M possesses necessarily a Pythagorean-Riemannian metric.

Weyl presents a long and difficult proof which establishes that the Postulate of Freedom and the requirement that the metric uniquely determines the affine connection together entail the existence of a unique non-degenerate quadratic form which remains invariant under infinitesimal rotation.[38]

The basic strategy is this. Starting with the assumption that M is Riemannian one shows that the Γ^i_{jk} are uniquely determined and that the n^3 coefficients Λ^i_{jk} for congruent transport are of the form

$$\Lambda^i_{jk} = A^i_{jk} + \Gamma^i_{jk},$$

where for each k, A^i_{jk} belongs to the Lie algebra of O(n). There are therefore $n^2(n-1)/2$ components for A^i_{jk} and $n^2(n+1)/2$ components for $\Gamma^i_{jk} = \Gamma^i_{kj}$. Consequently, since only the nature of g_{ij} is fixed (Riemannianness) there are

$$n^3 = n^2(n-1)/2 + n^2(n+1)/2$$

choices for Λ^i_{jk}. The nature of the metric does not restrict congruence transport in any way. Since for each k, A^i_{jk}

is a Lie algebra element of O(n), $A^i_{ik} = 0$ and is volume preserving.

To establish the converse, that is to show that the metric is Pythagorean in form, provided that a certain number of conditions are satisfied, one assumes nothing about the notion of rotations at each point. The only requirement is that they form a (microsymmetry) group and are volume preserving.

We assume that whatever the nature of the metric is, that is whatever its form might be, the class of possible metrics is such that no constraint is placed on congruent transport. That is, the class of admissable metrics is such that an arbitrary congruent transport

$$d\xi^i = \Lambda^i_{jk}\xi^j dx^k$$

constitutes a congruent transport for at least one member of the class.

We assume further that for any particular metric of the class, the set ~~of its~~ congruent transports $\{\Lambda^i_{jk}\}$ contains one and only one congruent transport Λ^i_{jk} which is an affine congruence transport and is denoted by Γ^i_{jk}.

For each Λ^i_{jk} in $\{\Lambda^i_{jk}\}$ define $A^i_{jk} = \Lambda^i_{jk} - \Gamma^i_{jk}$. Since volume is preserved it follows that $A^i_{ik} = 0$. Moreover, since Γ^i_{jk} is unique, $A^i_{jk} \neq A^i_{kj}$.

It follows that for each k, the set $\{\Lambda^i_{jk} - \Gamma^i_{jk}\} = \{A^i_{jk}\}$ forms a Lie algebra of maximal dimension $N = n(n-1)/2$, since $\Lambda^i_{jk} = A^i_{jk} + \Gamma^i_{jk}$ and $n^3 = nN + n^2(n+1)/2$.

THEOREM: Given the quantities A^i_{jk} such that

(a) for each k, the A^i_{jk} form a Lie algebra of dimension $n(n-1)/2$

(b) $A^i_{jk} \neq A^i_{kj}$ (unless $A^i_{jk} = 0$)

(c) $A^i_{ik} = 0$

then for each k, A^i_{jk} is skew in i and j and hence is an element of the Lie algebra of O(n), and the transport

$$d\xi^i = A^i_{jk}\xi^j dx^k$$

defines an infinitesimal linear mapping from the tangent space at x^k to the tangent space at $x^k + dx^k$ which leaves invariant a specific non-degenerate quadratic form.

Weyl's group theoretical formulation suggests an intuitive contrast between Euclidean "distance-geometry" and the "near-geometry" or field-geometry" of Riemann, by comparing the former to a crystal "build up of uniform unchangeable atoms in the regular and unchangeable arrangement of a lattice," and the latter to a liquid, "consisting of the same indiscernible unchangeable atoms, whose arrangement and orientation, however, are mobile and yielding to forces acting upon them."[39]

The <u>nature</u> of the metric field, that is the <u>nature</u> of the metric at p is <u>one</u> and therefore absolutely determined. It reflects, according to Weyl, the a priori structure of space or spacetime. In contrast, what is a posteriori, that is, accidental and capable of continuous change being causally dependent on the material content filling space, is the mutual orientation of the metrics at different points.

Weyl therefore suggests that spacetime possesses an a priori structure, but that the demarcation between the a priori and the a posteriori has shifted. Euclidean geometry is still preserved for the infinitesimal neighborhood of any given point but the coordinate system in which the metrical law assumes the standard form $ds^2 = \sum_{i=1}^{n} (dx^i)^2$, and which is characteristic for the orientation of the metric, is in general different from place to place.

In the context of his group-theoretical analysis Weyl makes the following interesting statement:

> I remark from an epistemological point of view: It is not correct to say that space or the world [spacetime, H.K.] is in itself, prior to any material content, merely a formless continuous manifold in the sense of analysis situs; the nature of the metric [its infinitesimal Pythagorean character, H.K.] is characteristic of space in itself, only the mutual orientation of the metrics at the various points is contingent, a posteriori and dependent on the material content. 40

Having incorrectly attributed to Weyl his pre-dynamical thesis of the relationality of the metric, (OI), it is not surprising that Grünbaum should find this statement somewhat out of line and feel the need to comment on it, saying:

> Does Weyl's phrase 'vor aller materiellen Erfüllung' indicate that he countenances an actually matter-empty space-time which is endowed with a determinate particular metrical structure and that he is thereby contradicting his seeming explicit espousal of OI ...? It seems to me that Weyl's phrase admits quite reasonably of a reading which avoids such a contradiction, if it is construed in its context as having the force of saying the following: apart from the contingent and a posteriori specifics of the material content, it is guaranteed a priori that the general form of any metric physically realised by matter will be infinitesimally Pythagorean. But this fully allows that in any one particular instantiation of this general form -- which can be instantiated by any one of an infinitude of incompatible metric structures -- material entities play the constitutive role assigned to external metric standards by Riemann's OI.[41]

Before we evaluate Grünbaum's commentary, let us first consider that according to Grünbaum's pre-dynamical thesis of the relationality of the metric, the spacetime of a matter-empty world would have to carry a pre-geometrical gravitational field. Grünbaum argues in fact that no physical significance could be attached to the claim that this field has an autonomous metric structure. In the case of empty spacetime, devoid of any members of the family of concordant standards (FCS), there cannot exist any difference in physical content between the characterization of the field as an autonomous metric field and its characterization as being pre-geometrical. To call such a field a metric field would amount to no more than "verbal baptism" and would be void of any physical significance Grünbaum tells us. Moreover, he says that counterfactual appeal to hypothetical congruence standards should not be countenanced, since within the context of General Relativity such entities must be excluded on account of their modifying effect on the field.

It would seem then, that in the case of a matter-empty world the relationalist must confine himself to a metric-free interpretation of the gravitational field. However, Grünbaum's reason for excluding hypothetical standards of congruence, such as atomic clocks for example, is unnecessarily restrictive. We do make approximations in General Relativity inspite of its non-linear character; in cosmology we even ignore planets and almost anything short of galaxies. Moreover, it would appear that the claim that "the g_{ij}-field is metric in a matter-empty spacetime devoid of FCS," can be regarded as a physically redundent and epistemically problematic claim only from the standpoint of some form of "homocentric ontology of operationalism," a position that is inimical to Grünbaum's basic intentions. If it is admitted that the gravitational field has the character of being autonomously metric in a non-empty spacetime, then it does not seem at all redundent physically, to consider it also as a metric field in the absence of matter, if it is conceived of as a rest or zero field. The situation here is not essentially different from other theoretical limiting cases in physics.

Let us return to Weyl's statement. It expresses the view that within the context of General Relativity the metric field cannot be removed such that empty spacetime remains. That is, Weyl expresses the impossibility of empty spacetime in the sense of understanding "empty" to mean not merely empty of all *matter* but also empty of all *fields*. At another place Weyl says:

> Geometry unites organically with the field theory; space is not opposed to things (as it is in substance theory) like an empty vessel into which they are placed and which endows them with far-geometrical relationships. No empty space exists here; the assumption that the field omit a portion of the space is absurd.[42]

Just as the electromagnetic field does not cease to exist but is in a state of rest in the portion of space where its value is zero, so, likewise, the metric field would be in a state of rest in a matter-empty world. As a *rest* or *zero* field it would possess the property of *metric homogeneity*; the mutual orientations of the orthogonal groups characterizing the Pythagorean metric everywhere would not differ from point to point. This means that in a matter-empty universe the *metric connection* is fixed. *Consequently, the set of congruence relations on spacetime is uniquely determined*.

Since the metric uniquely determines the affine connection, the homogeneous *metric* field (rest field) determines an integrable affine structure. Therefore, a flat Minkowsky spacetime consistent with the complete absence of matter, is endowed with an integrable connection and thus determines all (hypothetical) free motions. (The above analogy therefore breaks down in the sense that in the case of a zero electromagnetic field all physical effects would vanish whereas this is not the case for a zero g_{ij}-field).

According to Weyl then, there exists in the absence of matter a homogeneous metric field, a structural field (Struckturfeld) which has the character of a rest field, and which constitutes an all pervasive background that cannot be eliminated. The structure of this rest field determines the extension or reference of the space-time congruence relations and determines Lorentz invariance locally. The field possesses no net energy and makes no contribution to curvature.

The relation to Helmboltz and Lie is this: Both required homogeneity and isotropy for physical space. From a general Riemannian standpoint the latter characteristics are valid only for a matter-empty universe. Such a universe is flat and Euclidean, whereas a nonempty universe is inhomogeneous, anisotropic and of variable curvature.

Grünbaum's question whether Weyl "countenances an actually matter-empty space-time which is endowed with a determinate particular metric structure" must definitely be answered in the affirmative. From what has been said so far it is clear that Weyl is not "contradicting his seeming explicit espousal of OI", because Weyl never held such a view.

Moreover, it is incorrect on Grünbaum's part to interpret Weyl's statement as merely having the force of saying that "apart from the contingent and a posteriori specifics of the material content, it is guaranteed a priori that the general form of any metric physically realized by matter will be infinitesimally Pythagorean." Given the overall context of Weyl's group-theoretical characterization of metric structures and his pertinent comments elsewhere (as documented in Section 2) which express his realist and anti-relationalist attitude with regard to spacetime structure, it is quite clear that Weyl's statement must be interpreted here in the way I have suggested, namely, as an explicit denial of Grünbaum's pre-dynamical thesis of the relationality of the metric.

But apart from these considerations one might want to ask why Grünbaum does not find it problematic that the nature of the metric, its Pythagorean form, is given once and for all independent of the material content of spacetime and hence independent of any ontological constitutors belonging to FCS. Why is it, that the bona fide physical g_{ij}-field of matter-empty spacetime is to be regarded as somehow autonomous with respect to the nature of the metric? More specifically, why is it that the g_{ij}-field is not autonomously metric -- in the sense of not being autonomously metric with respect to one particular instantiation of the Pythagorean metric form -- and yet it qualifies as having the character of being autonomously Pythagorean?

Clearly, the <u>nature</u> of the metric is not dictated by, nor is it canonically definable, in terms of the local differential topological structure. Now it is quite true that <u>any</u> differentiable manifold admits a Riemannian (though not necessarily pseudo-Riemannian) structure. But it does not follow therefore that the <u>physical</u> g_{ij}-field has this type of metric structure also. Are there perhaps some a priori grounds for assuming that the g_{ij}-field has to be Riemannian? It is incorrect when Grünbaum says:

> ... it appears from Weyl's various writings that he claims to be able to deduce a priori -- in the manner of the presuppositional method employed by Kant's transcendental deduction of the categories -- that physical space (-time) as such is endowed with an infinitesimally Pythagorean metric.[43]

Although Weyl does use a notion of 'a priori' and refers also to Kant in this context, he nowhere suggests that he deduces a priori in the manner of Kant's method that physical space is endowed with the Pythagorean form of the metric. It will not be necessary here to clarify Weyl's special use of 'a priori' nor his reference to Kant in order to show that he is definitely not doing what Grünbaum interprets him to be doing. Weyl showed that if a manifold is such that among all possible systems of parallel displacements there exists one and only one which is at the same time a system of congruent displacements, then M possesses necessarily a Pythagorean metric. Now the antecedent condition is an additional postulate, which, as Weyl points out, goes beyond what lies in the concepts metric, metric connection and parallel displacement. Weyl then says:

> I now come to the 'synthetic part' in Kant's sense. Among the different types of metric spaces we shall characterize through a ~~simple~~ intrinsic property that one, to which, according to Pythagoras and Riemann, real space belongs.[44]

And in another place Weyl says much the same:

> I now come to the synthetic part in Kant's sense. The task is to give a precise formulation of the postulate, alluded to earlier, which determines the type of rotational group characteristic of the world.[45]

The postulate Weyl is referring to is identical in content to the above stated antecedent condition. This postulate is certainly suggestive and reasonable. It emerged as a fundamental fact in the construction of Relativity Theory and Riemannian geometry. Weyl showed that this postulate, together with the Postulate of Freedom, constitutes the necessary and sufficient condition for the uniqueness of the Pythagorean metric. Thus, while this postulate is certainly compelling and suggestive from the point of view of Riemannian geometry and the theory of General Relativity, it does not constitute an a priori necessary and sufficient condition for the uniqueness of the Pythagorean metric. It is conceivable to think of metrics, such that, with respect to them, it is not the case that there exists one and only one affine parallel displacement that is at the same time a congruent displacement. Consequently, it is conceivable that one and the same local differential topological structure may admit metrics that either do or do not uniquely determine the affine connection.

In sum, the claim for the uniqueness of the Pythagorean form of the metric cannot be established on the basis of the local differential topological structure alone, nor can it be argued for in some a priori manner. We are therefore still left with the unanswered question: Why is the g_{ij}-field of matter-empty spacetime not autonomously metric with respect to one particular instantiation of the Pythagorean metric form although it qualifies according to Grünbaum, as having the autonomous character of being pseudo-Riemannian?

Although Einstein was not directly influenced by Riemann, he helped to bring Riemann's ideas to the fullest concrete realization by extending them into a physical theory of gravitation, and developed the laws of gravitation, according to which the matter content of spacetime determines the metrical structure.

Riemann did not foresee Einstein's theory of General Relativity. There is no indication in Riemann's own work on gravitation and electromagnetism that would suggest that he anticipated the conceptual revolution concerning our understanding of space and time implied by Einstein's theory. Restating an earlier remark, we might say, however, that Riemann foresaw something like its possibility in the following sense:

By formally separating the non-topological (or more accurately the post-topological structures such as the affine, projective, conformal and metric structures) from the manifold, so that these structures are no longer rigidly tied to it, Riemann deprived them of their formal geometric rigidity and, on the basis of his infinitesimal geometric standpoint or "near-geometry," allowed for the possibility to interpret them as mathematical representations of flexible, dynamical, physical structural fields on the manifold of physical spacetime, geometrical fields which reciprocally interact with matter.

FOOTNOTES

1. [1], pp. 495, p. 500; [2], p. 314, pp. 321-323.
2. [17], p. 87.
3. [17], p. 96.
4. [17], p. 86.
5. [16], in [18], Vol. III, p. 338.
6. [12], in [18], Vol. II, p. 339.
7. [16], in [18], Vol. III, p. 337.
8. [16], in [18], Vol. III, pp. 338-339.
9. [7], in [18], Vol. II, p. 2.
10. [17], p. 113.
11. [15], in [18], Vol. II, 498.
12. See Section A1.
13. [16], in [18], Vol. III, p. 336.
14. [17], p. 78.
15. [2], p. 321.
16. [2], pp. 344-345.
17. [2], p. 345.
18. [2], p. 345.
19. See Section 4.7 and the end of Section 5.4.
20. [10], in [18], Vol. II, pp. 261-262.
21. [4], pp. 76-77.
22. [9], in [18], Vol. II, p. 67.
23. [19], p. 299.
24. [19], p. 302.
25. [7], in [18], Vol. II, p. 2.

26. [7], in [18], Vol. II, p. 2.

27. [14], p. 13.

28. See Section A6 for a precise characterization of a directing field.

29. [19], p. 219.

30. [19], p. 222.

31. The discussion of the first approach follows [5] and [6].

32. [11] and [14].

33. [8], in [18], Vol. II.

34. [8], in [18], Vol. II, p. 30.

35. [19], p. 123.

36. [19], p. 124.

37. [14] and [13], in [18], Vol. II.

38. A modern statement of the theorem is given in [3], pp. 288-289.

39. [17], pp. 89-90.

40. [13], in [18], Vol. II, p. 266.

41. [2], p. 369.

42. [17], p. 172.

43. [2], p. 369.

44. [19], p. 139.

45. [14], p. 49.

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CHAPTER 5

CONSTRUCTIVE AXIOMATICS

5.1 INTRODUCTION

A great deal of literature on the status and meaning of the Laws of Inertia in spacetime theories has nurtured and given wide currency to the claim that the laws are conventional in character, that they are definitions, or circular and without empirical content.

Philosophers who argue for the conventional character of the laws, do so, either emphasizing epistemological or ontological considerations concerning the structure of spacetime.

Those who argue for their conventional character mainly on epistemic grounds, point out, that the laws do not supply independent criteria of what is to count as force-free or natural motion. The only way of knowing when no forces act on a body is that it moves as a free particle travelling along the geodesics of spacetime. But how, without already knowing the geodetic structure of spacetime, is one to determine which particles are free and which are not? And of course to find the geodesics of spacetime one must use free particles.

Those philosophers who argue in support of the conventional character of the laws mainly from ontological considerations concerning the nature of spacetime structure advance the view that spacetime is structurally amorphous in the sense that what counts as a standard of no-acceleration or uniform motion is not dictated by a physically real, causally efficacious dynamic structure of spacetime. Rather, forces are defined relative to a conventionally chosen standard of no-acceleration and depend for their existence on that conventional choice.

Recently the problem concerning free or natural motion has become a pressing issue within the particular context of the Constructive Axiomatics for General Relativity Theory (GTR). In a paper entitled "The Geometry of Free Fall and Light Propagation", Ehlers, Pirani and Schild (EPS) propose a set of constructive axioms which are propositions about the local incidence and differential topological properties of arbitrary massive particles, freely falling massive particles and light propagation.[1] The constructive axioms are propositions about a few general qualitative assumptions concerning free(fall)motion and light propagation that can be verified directly through experience in a way that does not presuppose the full blown edifice of GTR. From these axioms EPS generate a unique pseudo-Riemannian metric, unique up to a constant positive factor.

One of the constructive axioms employed by EPS -- the Projective Axiom -- is a statement of the infinitesimal version of the Law of Inertia, the Law of free (fall) motion which contains Newton's First Law as a special case in the absence of gravitation. The problem is how to introduce a class of preferred motions. How is one to characterize that particular path structure which would be realized by a free particle, namely a neutral, spherically symmetric non-rotating test particle, and avoid the circularity problem surrounding the notion of a free particle?

Since EPS do not provide an independent non-circular criterion with which to characterize free (fall) motion, their approach has been charged with circularity by philosophers such as Grünbaum, Salmon, Sklar and Winnie. One has argued essentially that any criterion that determines which bodies are suitable as freely falling test bodies and permits their identification, presupposes the specification of geometric structures beyond those implied by the local differential topologioal structure of spacetime.

In Section 2 we shall briefly discuss the nature of the constructive axiomatic approach and show it to be a natural outgrowth of Weyl's realist field ontology of geometric structure. Section 3 contains a brief discussion of Weyl's approach to the causal-inertial method and is followed with a brief outline of EPS's improved version of Weyl's treatment.

We shall follow EPS's overall approach, but the notation and the organization of the axioms will at some places be slightly modified and improved upon. Detailed proofs, however, will not be reproduced and we refer the reader to the original sources.

5.2 THE CAUSAL-INERTIAL METHOD

Einstein suggested the distinction between principle theories and constructive theories.[3] The aim of a constructive theory is to reduce a wide class of diverse complex physical processes to simpler ones. We claim to understand complex physical processes when we appreciate the manner in which such processes are reducible to or constructed out of simpler ones. For example, the kinetic theory of gases constructs mechanical, thermal and diffusional processes from the hypothesis of molecular motion. A principle theory, on the other hand, postulates abstract structural constraints which events are held to satisfy. Einstein's example is the classical theory of thermodynamics.

The Special and General theories of Relativity are principle theories of spacetime structure. The four-dimensional pseudo-Riemannian metric manifold is the mathematical model of the physical spacetime of GTR. Weyl distinguished between two simpler more primitive structures of that model: The conformal structure defined by the field of null cones and the projective structure of paths defined by the set of all unparametrized geodesics. Weyl was the first to introduce the geodesic or causal-inertial method by suggesting that the conformal (causal) structure associated with light propagation, together with the projective structure associated with free (fall) motion, uniquely determine the spacetime metric of GTR.

Whereas Weyl emphasized from a physical point of view the roles of light propagation and free (fall) motion in determining the causal and projective structures of spacetime respectively, he did not use these more primitive structures directly ~~in order to de~~rive from them and their compatibility relation the existence of a unique affine connection (Weyl connection, c.f. Section 5.4). He rather considered the affine and metric structures as fundamental in his mathematical analysis and saw the projective and conformal structures as arising from them by abstraction. Thus, in his group theoretical justification of the uniqueness of the Pythagorean-Riemannian metric, Weyl considered the congruent mappings of a tangent space onto itself, and the translation of a tangent space at some point to a neighboring point, as fundamental operations (c.f. Section 4.9).

Using the conformal and projective structures directly, EPS have constructed an improved version of Weyl's geodesic method and derive a unique pseudo-Riemannian spacetime metric solely as a consequence of a set of natural, physically well motivated constructive, 'geometry-free' axioms concerning the incidence and differential topological properties of light propagation and free fall.

The 'geometry-free' axioms are propositions about a few general qualitative assumptions concerning free (fall) motion and light propagation which can be verified directly through experience in a way that does not presuppose the full blown edifice of the theory of General Relativity. From these axioms, the theoretical basis of the theory is reconstructed step by step. Following Reichenbach,[4] EPS call their approach constructive axiomatics.

For the most part, physical theories are presented from the deductive axiomatic point of view; that is, the exposition begins with a set of postulates concerning the existence of high level structures and proceeds by logical deduction to lower level phenomena which may be directly confronted by experiment. In the case of GTR, the assumed existence of a pseudo-Riemann (or Lorentz) structure leads to the existence of affine, projective and conformal structures which govern the behavior of massive particles and light rays. For example, in Synge's chronogeometric approach one adopts standard clocks and freely falling particles as primitive physical concepts. One assumes the metrical structure as fundamental by defining a chronogeometric tensor g_{ij} on the manifold, such that the distance between two nearby spacetime events, contained in a one-dimensional submanifold (worldline) of a standard clock, equals the time interval measured by that clock.

This approach has the advantage of making clear at the outset the ontological commitments involved and of providing a logically compelling understanding of how the lower level phenomena are explained by the theory. However, direct contact with the higher level structure is frequently elusive. The axioms within the deductive approach, are quite removed from the level of direct experience, and, while they may be adequate for the explanation of lower phenomena, they may not be necessary. For example, the chronogeometric axioms for GTR suffice for the explanation of the phenomena explained by the theory; but it is conceivable that a Weyl structure would be sufficient. Thus within the context of a deductive axiomatic approach, only indirect and probable confirmation of the axioms representing higher level abstractions is possible through the direct confirmation of their consequences.

The constructive axiomatic approach is the reverse of the deductive axiomatic procedure. The constructive axioms deal with directly observable phenomena at as low a level as possible. The aim is to formulate axioms which may be directly confronted by experiment, and then deduce from these low level axioms the existence of higher level structures. As Reichenbach remarks, the constructive axioms

> have been chosen in a way that they can be derived from the experiments by means of pre-relativistic optics and mechanics. All are facts that can be tested without the use of the theory of relativity . . . The particular factual statements of the theory of relativity can all be grasped by means of pre-relativistic conceptions; only their combination within the conceptual system of a theory is new.[5]

The aim of a constructive axiomatic approach to a principle theory of spacetime structure is to exhibit the physical basis for the particular structural constraints which the principle theory postulates.

The structures contained in the mathematical model of a principle theory should all have in principle a link to physical experience. Spacetime models with inherent struc-

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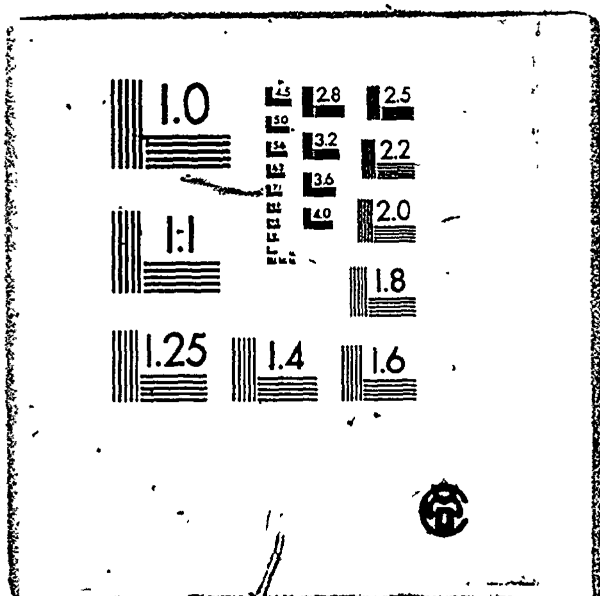


tures that do not relate to experience (e.g., absolute time) are defective for that reason. Hence, it must be theoretically possible, that is, possible in theory, to relate the various structures to experience in a way that is consistent with the theory.

A constructive axiomatic approach should therefore satisfy the basic requirement of any proper and complete theory. Completeness requires that the reconstruction of the various structures inherent in the mathematical model of a principle theory of spacetime be realizable by means of relatively simple physical systems that are themselves well defined within the specific theory being considered, that is, that can be considered as an interpretation of the inherent structures of the spacetime model and are consistent with the theoretical consequences of the theory which presupposes that model. Einstein was well aware of this problem and considered the use of clocks and rigid rods an undesirable makeshift.[6] Unlike light propagation and freely falling particles, rigid rods and ideal clocks are relativistically ill defined and are thus unsuitable for the determination of the inherent structures of the spacetime of general relativity. The concepts of a theory, its formulation and measuring devices should all lead to a unified, self-sufficient and conceptually coherent world picture.

The essential idea of the geodesic method is to discover through the behavior of physical systems various intrinsic primitive geometrical spacetime structures. It is in spirit analogous to Helmholtz's procedure of deducing the existence and form of the metric of physical space. We recall that Helmholtz asked: What must the geometric structure of space be in order that a mechanics of rigid bodies be realizable in that space? Thus Helmholtz is essentially asking what abstract structural constraints must a principle theory of mechanics postulate that certain events must satisfy. According to Helmholtz, the structure of space follows from the possibility of congruent transport of rigid bodies; that is, the structure of space constitutes a necessary condition for the possibility of the realizability of certain physical processes and operations within that space; in particular, whether or not space possesses a constant curvature, or whether space is a general Riemannian space depends on whether or not physics allows the introduction of ideal rigid bodies.

The structure of space is, according to Helmholtz, the framework for possible physical laws. Certain types of laws presuppose certain types of spaces. Hence, on this view, the Law of Inertia presupposes a unique projective structure and is to be regarded as an empirical assertion about that structure.

The conventionalist view that considers the behavior of material entities as being ontologically constitutive of the metrical structure of spacetime is clearly at variance with the notion of a principle theory. It is clear that the views of Weyl and Helmholtz are directly opposed to those of geometrical conventionalism. According to Weyl we discover through the behavior of physical phenomena an already determined metrical structure of spacetime.

The association of the projective and conformal structure of spacetime with the paths defined by the set of all unparametrized geodesics that are realized by free particles and light rays does not involve the claim that the world is filled with free particles and light rays everywhere. We do not mean, for example, that every timelike geodesic is the realization of some actual particle in free (fall) motion. Rather, we think of timelike geodesics as the set of spacetime locations which are the locations of actual or possible events constituting the history of particles in free (fall) motion.

A suitable test particle provides us with epistemic access to the electromagnetic field strength at some point. The electromagnetic field strength is revealed by, not defined by a test object. It is not ontologically reducible to nor does it depend relationally on the actual presence of test particles.

In an analogous way, the various spacetime structures, that is, the various geometrical structural fields which are postulated by the principle theories of spacetime structure, are not ontologically reducible to nor relationally dependent upon the actual existence of material systems (such as for example, free particles and light rays) and the relations between them. For example, what is often overlooked is that optics and electrodynamics played essentially only a heuristic role in the construction of Special Relativity. The latter's justification does not depend in an essential way on the actual existence of light. Properly understood, the velocity of light is an invariant upper limiting velocity. Special relativistic mechanics says that only objects with zero rest mass possess an upper limiting velocity. It does not say that there exist such objects in nature. The causal structure postulated by Special Relativity is not ontologically reducible to nor does it depend relationally on the actual existence of photons or other possible objects with vanishing rest mass. The existence of light is a gift of nature which certainly aided in the construction of Special Relativity; but it is not necessary for its validity.

5.3 WEYL'S GEODESIC METHOD

In a letter to Felix Klein which was published in 1921 under the title "Zur Infinitesimalgeometrie: Einordnung der projektivén und konformen Auffassung",[7] Weyl proved the important theorem that the projective and conformal structure of a metric space determine the metric uniquely.

The essential reasoning of Weyl's proof is this. Suppose we are given a metric g on M. Then such a metric determines an equivalence class of conformally equivalent metrics, namely,

$$[g] = \{\bar{g} | \bar{g} = e^{\theta} g\}.$$

By the fundamental principle of infinitesimal geometry (c.f. Chapter 4, Section 7) g determines the affine structure Γ of M uniquely. Therefore, under a conformal transformation $g \to e^{\theta} g = \bar{g}$ the change of the components of the affine connection is given by equation 4.8.2, namely

$$(\bar{\Gamma} - \Gamma)^{i}_{jk} = \tfrac{1}{2}(\delta^{i}_{j}\theta_{k} + \delta^{i}_{k}\theta_{j} - g_{jk}\theta^{k}). \tag{3.1}$$

Weyl shows that such a change preserves the projective structure of M (compatibility between the projective and conformal structure) if and only if $\theta = 0$.

Suppose a change of the metric field preserves the conformal structure. Then (3.1) must hold. If such conformal change is compatible with the projective structure of M, then we also require (see Section 6, Chapter 4) that the equation

$$(\bar{\Gamma}^i_{jk} - \Gamma^i_{jk})X^jX^k \propto X^i \tag{3.2}$$

is satisfied. From the above expression and (3.1) one obtains

$$\tfrac{1}{2}(\delta^i_j\theta_k + \delta^i_k\theta_j - g_{jk}\theta^i)X^jX^k \propto X^i. \tag{3.3}$$

Therefore,

$$X^i(\theta_jX^j) - \tfrac{1}{2}\theta^i(g_{jk}X^jX^k) \propto X^i.$$

Since $X^i(\theta_jX^j) \propto X^i$, we merely require to show that

$$(g_{jk}X^jX^k)\cdot\theta^i \propto X^i \tag{3.4}$$

in order for (3.3) to be satisfied. We now assume that $X^i, Y^i \varepsilon TM_{p_\circ}$, such that

$$\{\lambda X^i | \forall\lambda\varepsilon\mathbb{R}\setminus\{0\}\} \cap \{\lambda' Y^i | \forall\lambda'\varepsilon\mathbb{R}\setminus\{0\}\} = \emptyset,$$

that is, X^i lies in a different direction than Y^i at p. We further assume that

$$g_{jk}X^jX^k \neq 0; \quad g_{jk}Y^jY^k \neq 0.$$

Consequently

$$\theta^i = k\left(\frac{1}{g_{jk}X^jX^k}\right)X^i = k'\left(\frac{1}{g_{jk}Y^jY^k}\right)Y^i.$$

However, X^i and Y^i are linearly independent; consequently, it is necessary that $k = 0 = k'$ and hence $\theta^i = 0$. This completes the proof.

As we shall discuss in the next section, EPS's compatibility requirement is weaker than Weyl's; that is EPS only require that the <u>Projective null geodesics</u>

$$\frac{d^2x^i}{dt^2} + \Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \mu(t)\frac{dx^i}{dt}$$

and <u>conformal null geodesics</u>

$$\frac{d^2x^i}{dt^2} + K^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \tilde{\mu}(t)\frac{dx^i}{dt}$$

are compatible. In other words,

$$(\Pi^i_{jk} - K^i_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt} = (\mu(t) - \tilde{\mu}(t))\frac{dx^i}{dt}$$

determines an equivalence class of solutions $[x^i(t)]$ by requiring only that

$$g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

Weyl on the other hand employs a stronger compatibility requirement by assuming

$$g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \neq 0.$$

With this result Weyl established the <u>geodesic method</u> with which to determine the spacetime metric. Weyl says;

> ... if in the real world it is possible for us to discern causal propagation, and in particular light propagation, and if moreover, we are able to recognize and observe as such the motion of free mass points which follow the guiding field, then we are able to read off the metric field from this along, without reliance on clocks and rigid rods.[8]

The motions of freely falling particles which follow the guiding field (or geodesic directing field) <u>reveal</u> the geodesics of spacetime, that is, the geodesics corresponding to a symmetric connection. A manifold for which a special class of curves has been singled out, namely, the geodesics, possesses a projective structure. Conversely, a manifold has a projective structure if it is endowed with an equivalence class of projectively related affine connections.

The causal propagation or light propagation <u>reveals</u> at each point of spacetime the infinitesimal light cone. A manifold in which the null cone is singled out in the tangent space TM_p for all $p \epsilon M$, has a conformal structure or causal field defined on it. Conversely, a manifold M has a conformal structure if it is endowed with an equivalence class of conformally equivalent metrics of Lorentz signature, which are proportional to one another and which define the same null cones, that is, which are defined by the relation

$$\bar{g} = e^{\theta} g$$

where θ is any positive real valued function on M.

5.4 THE CONSTRUCTIVE AXIOMS FOR GENERAL RELATIVITY

EPS adopt as primitive concepts the notion of event, particle path and light ray. The constructive model of spacetime is based on a triple of sets $\langle M, P, L\rangle$ of the objects corresponding to these primitive concepts.

It is assumed that the set $M = \{e, p, q, p_1, q_1, \ldots\}$ of events has a topology which is Hausdorff and has a countable basis. This assumption is neutral with respect to the various geometric structures whose existence will be constructively established. No particular global character is assumed for the topology. The sole purpose in assuming a topology is to permit the clear statement of local axioms through the use of such terms as neighborhood. Moreover, the differential structure is not presupposed.

The elements of the set of actual or possible events of M represent actual or possible locations in spacetime.

The members of the sets $P = \{P, Q, P_1, Q_1, \ldots\}$ and $L = \{L, N, L_1, N_1, \ldots\}$ are subsets of M which represent the possible or actual paths of massive particles and light rays in spacetime. For brevity they will be called particles and light rays respectively.

The set of constructive axioms constitute physically motivated and directly testable assumptions about the behavior of light rays and massive particles. With the first six axioms a differential manifold structure is introduced on the set M that is sufficient for the localization of events by means of local coordinates (radar coordinates). Once M is given a differential manifold structure through the introduction of local radar coordinates by means of particles and light rays (such that any two radar coordinates are smoothly related to one another) one can do calculus on M and one may speak of tangent and direction spaces.

With further axioms one introduces the conformal and projective structures on M. Finally, an axiom of compatibility between these two structures yields a Weyl space. M is now endowed with a unique symmetric affine connection. A reduction to a pseudo-Riemannian structure involves setting Weyl's length-curvature tensor equal to zero.

It should be noted that when one says in this context that additional structures will be introduced on M, one does not mean that these additional structures will be operationally defined on M. Rather, as should be apparent from the preceding discussion, the constructive axioms are directly testable propositions about the behavior of massive particles and light propagation and are statements about the structures of spacetime revealed by and not operationally defined by the behavior of these probative physical systems.

RADAR COORDINATES AND LOCAL DIFFERENTIAL TOPOLOGICAL STRUCTURE

It is important to emphasize that the members of P represent possible or actual paths of arbitrary massive particles which may have some internal structure such as higher order gravitational and electromagnetic multipole moments and which may therefore interact in complicated ways with various physical fields. In order to constructively establish the projective structure of spacetime it is necessary to single out a subset of P, namely P_f, the set of possible or actual paths of spherically symmetric, electrically neutral particles (that is, the world lines of freely falling particles).

However, the subset P_f, $P_f \subset P$, cannot be properly characterized until a coordinate system (differential structure) is available. Consequently, it is necessary to employ arbitrary particles in the statement of those axioms which lead to the local differential structure of spacetime.

All physical observations are local in character. Consequently, it is necessary to formulate the constructive axioms in such a manner that only local experiments are required to test them, and to treat global questions as extension problems. In fact, some of the axioms, if stated globally, would be false in some possible spacetime models.

Axiom 1: For every event $p \in M$, there exists at least one $P \in \mathcal{P}$ such that $p \in P$.

Remark: When we speak of events, particles and light rays, we always have in mind actual or possible events, particles or light rays.

Intuitively, particles exist continuously, and, except possibly for quantum processes, changes occur with some degree of smoothness. Since the treatment presented is limited to purely classical processes we assume the next axiom.

Axiom 2: For all $P \in \mathcal{P}$, P is a C^∞ one-dimensional manifold (path) diffeomorphic to $\mathbb{R}$.

Remark: Particles may be thought of as idealized observers. A local coordinate on a particle is interpreted as a sense of time as shown by a non-metric, possible irregular clock associated with the particle. The assumption of a C^∞-differentiable structure for each particle eliminates the necessity of keeping track of differentiability classes. One could assume a C^k-differentiable structure for k large enough to ensure that the projective and conformal curvatures are C^0. It would suffice to assume C^5. Part a) of the next axiom restricts the size of the spacetime region sufficiently to eliminate global problems which arise in worlds with closed or compact spacelike sections.

or cyclic times. We want to guarantee that there are <u>at most</u> two light rays connecting q with P. Part b) restricts the spacetime region still further in order to guarantee the existence of <u>exactly</u> two light rays through q connecting q with P (existence of radar echo).

<u>Axiom 3</u>: For every event $p \varepsilon M$ there exist open neighborhoods $U_p \subset V_p \subset M$ such that

a) for every $P \varepsilon \mathcal{P}$, and for all events $q \varepsilon V_p$ such that $q \not\varepsilon P$, there exist <u>at most</u> two light rays through q such that their restrictions to V_p intersect $P \cap V_p$ in at most one point each.

b) for every $P \varepsilon \mathcal{P}$ such that $p \varepsilon P$ and for all $q \varepsilon U_p$ such that $q \not\varepsilon P$, there exist precisely two distinct light rays L_1, $L_2 \varepsilon \mathcal{L}$ through q such that the restrictions $L_1 \cap V_p$ and $L_2 \cap V_p$ each intersect $P \cap V_p$ in precisely one event, p_1 and p_2 respectively, such that $p_1 \neq p_2$.

Illustration for part a): q is connected in V_p by L_1, L_2 to P. q' is only connected in V_p to P by L_2'. q" is not connected in V_p to P by any light ray.

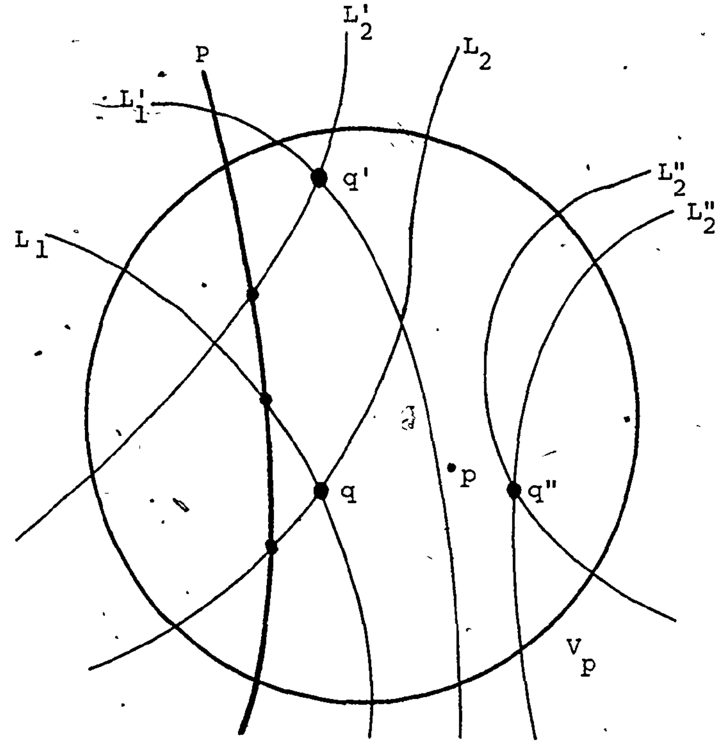
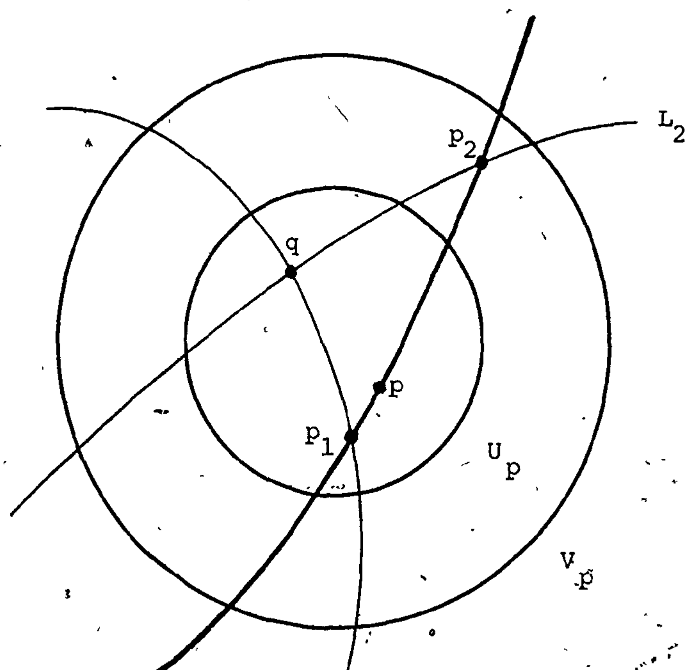


Illustration for part b): $p \varepsilon U_p \subset V_p$ and q is connected to p_1 and p_2 on P restricted to V_p by L_1 and L_2 such that L_1 and L_2 are restricted to V_p.



Remark: The empirical basis for this axiom is the fact that, at least for sufficiently small regions of spacetime, when an object is tracked by radar, there corresponds a unique echo to each emitted pulse.

Lemma: For any $p \in M$, let $U_p \subset V_p \subset M$ be neighborhoods which satisfy the conditions of axiom 3 and let P be any particle through p and let $<<$ be one of the two possible orderings on P (see remark after axiom 2). Then for all $Q \in \mathcal{P}$, such that $Q \cap U_p \neq \emptyset$, there exist two maps $f_\pm: Q \cap U_p \to P$, such that for all $q \in Q \cap U_p$, $f_-(q) << f_+(q)$.

Proof: By axiom 3, for every $q \in Q \cap U_p$, there exist precisely two distinct events on $P \cap V_p$ which are related to q by light rays. These events may be labeled $p_\pm$ and distinguished by $p_- << p_+$. Then the maps $f_\pm$ are defined by $f_\pm(q) = p_\pm$.

Remark: No 'arrow of time' is introduced here. Either of the two orderings on P will serve as well as the other.

On the basis of the axioms assumed so far, the maps $f_\pm$ are injective, but there is no guarantee that they will be surjective and hence bijective. The next axiom says that our communications are smooth.

<u>Axiom 4</u>: The maps $f_{\pm}$ are, with suitable target restrictions, diffeomorphisms.

<u>Remark</u>: Only the C^{∞}-differentiable structure of the particles is involved here.

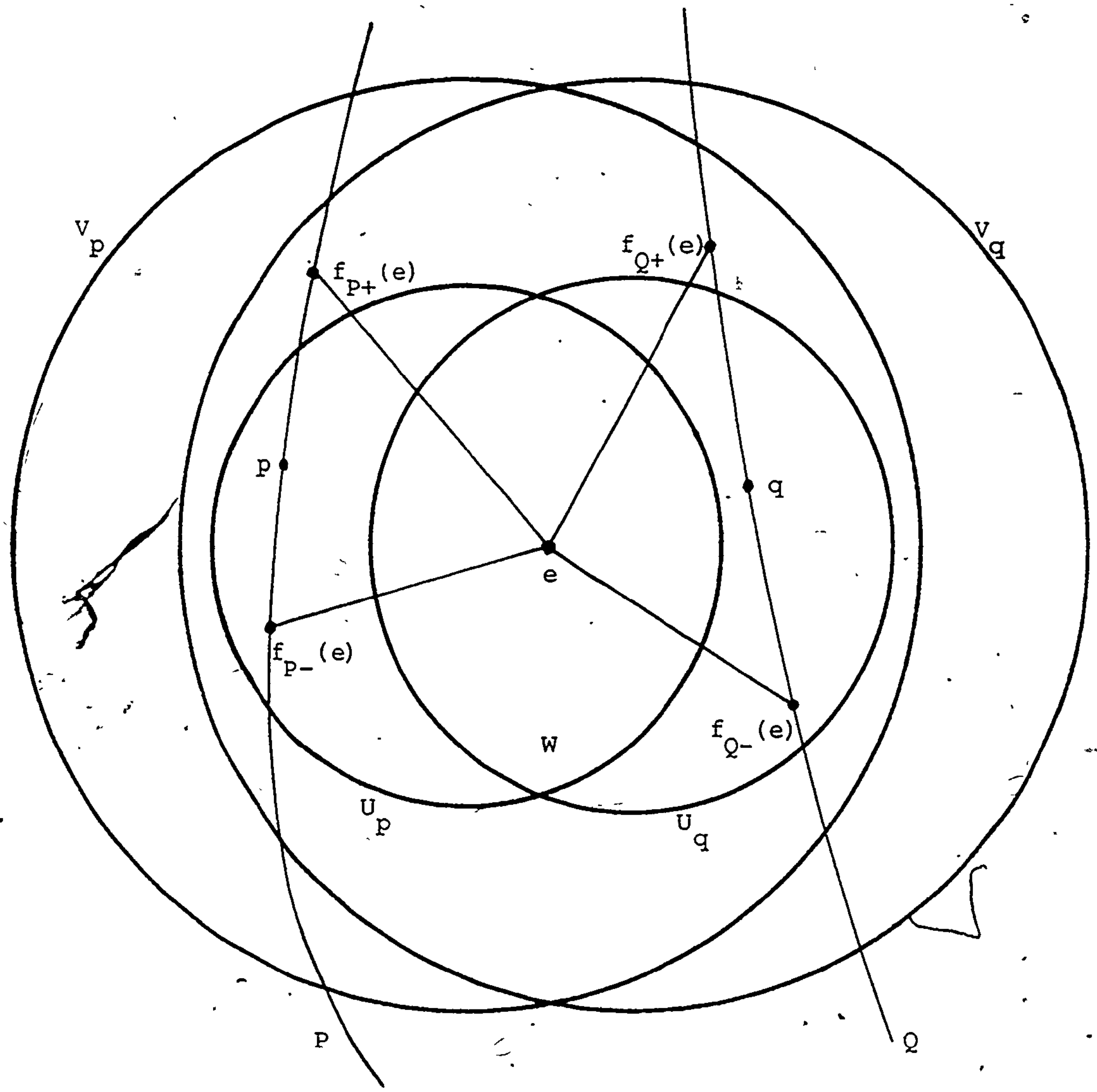
<u>Definition</u>: The composition maps $E_{+-}=f_{+}\circ f_{-}^{-1}$ and $E_{-+}=f_{-}\circ f_{+}^{-1}$ are called echo maps. Each is the inverse of the other. They are local diffeomorphisms of P.

<u>Lemma</u>: For any $p\epsilon M$, let $U_p \subset V_p \subset M$ be open neighborhoods which satisfy the conditions of axiom 3 and let P be any particle through p and let << be any of the two possible orderings on P. Then, there exist two maps $f_{p\pm}: U_p \to P \cap V_p$, such that for all $q\epsilon U_p$, $f_{-}(q) << f_{+}(q)$.

<u>Remark</u>: When restricted to particles, these maps are just the message maps defined above. Their definition is entirely similar to that of the message maps.

<u>Definition</u>: Let p and q be any two events with neighborhoods $U_p \subset V_p$ and $U_q \subset V_q$, respectively, which satisfy the conditions of axiom 3, and let P and Q be particles through p and q with time charts t_P and t_Q. If $W = U_p \cap U_q \neq \emptyset$, W is called a radar neighborhood, and the radar map $x: W \to \mathbb{R}^4$ is defined as follows: For all events $e\epsilon W$

$$x(e) = (t_P \circ f_{P+}(e),\ t_P \circ f_{P-}(e),\ t_Q \circ f_{Q+}(e),\ t_Q \circ f_{Q-}(e)).$$



Lemma: Radar neighborhoods and maps exist.

Proof: One need only take $p \varepsilon U_q$ or $q \varepsilon U_p$.

Remark: A pair (W,x) consisting of a radar neighborhood and a radar map may not yet be called a radar chart. It is necessary to ensure that the map $x: W \to \mathbb{R}^4$ is a homeomorphism onto an open set of $\mathbb{R}^4$. This amounts to the requirements that spacetime is four-dimensional and that the particles P and Q are not accidentally related in some special way, as coplanar lines are in Minkowski spacetime. Moreover, it is necessary to assume that every event has a suitable radar neighborhood with a radar map which together form a chart.

<u>Axiom 5:</u> For all $p \varepsilon M$, there exists a radar <u>chart</u> (W,x) with $p \varepsilon W$.

<u>Axiom 6:</u> If (W_1,x_1) and (W_2,x_2) are any two radar charts such that $W_1 \cap W_2 \neq \emptyset$, the transition maps $x_1 \circ x_2^{-1}$ and $x_2 \circ x_1^{-1}$ are local diffeomorphisms of $\mathbb{R}^4$.

<u>Remark:</u> It follows from axioms 5 and 6 that the set of all radar charts forms a C^∞-atlas for M: There exists a family

$$\mathcal{W} = \{\langle x_\alpha, W_\alpha, P_\alpha, Q_\alpha; V_\alpha\rangle\}$$

of quintuples with $W_\alpha \subset V_\alpha \subset M$, P_α, $Q_\alpha \varepsilon \mathcal{P}$, such that

i) x_α is a radar coordinate system with domain W_α, based on P_α, Q_α relative to V_α.

ii) $M = \cup_\alpha W_\alpha$, that is $\{W_\alpha\}_{\alpha\varepsilon I}$ covers M.

iii) for each pair (α,β), x_α, x_β are smoothly related. Any radar coordinate system $\langle x,W,P,Q;\ V\rangle$ for M is smoothly related to the x_α's. Note, that any other coordinate system which is diffeomorphic to a radar coordinate system is also permissible.

Axioms 5 and 6 together express that M can be given the structure of differential manifold. One may now speak of the tangent bundle T(M) of M. The fiber over $p\varepsilon M$ is the space $T(M_p)$ of tangent vectors at $p\varepsilon M$. By axiom 5, $\dim(M) = \dim T(M_p) = 4$.

A one-direction ξ_1 is a one-dimensional subspace of $T(M_p)$. Each non-zero vector $X\varepsilon T(M_p)$ determines a one-direction ξ_1. Any two vectors X, $Y\varepsilon T(M_p)$ determine the same one-direction ξ_1 if $X=kY$, with $k\varepsilon\mathbb{R}\setminus\{0\}$. The fibers $\mathbb{D}^1(M_p)$ of the bundle of 1-directions $\mathbb{D}^1(M)$ are the canonical projective spaces associated with the tangent spaces $T(M_p)$.

The next two axioms are required by EPS to prove certain theorems and are stated here mainly for completeness.

<u>Axiom 7</u>: For all $L\varepsilon\mathcal{L}$, L is a C^∞ one-dimensional manifold of M.

<u>Axiom 8</u>: If $f: P \to Q$ is a message map for an open subset of P to Q, then the one parameter family of light rays

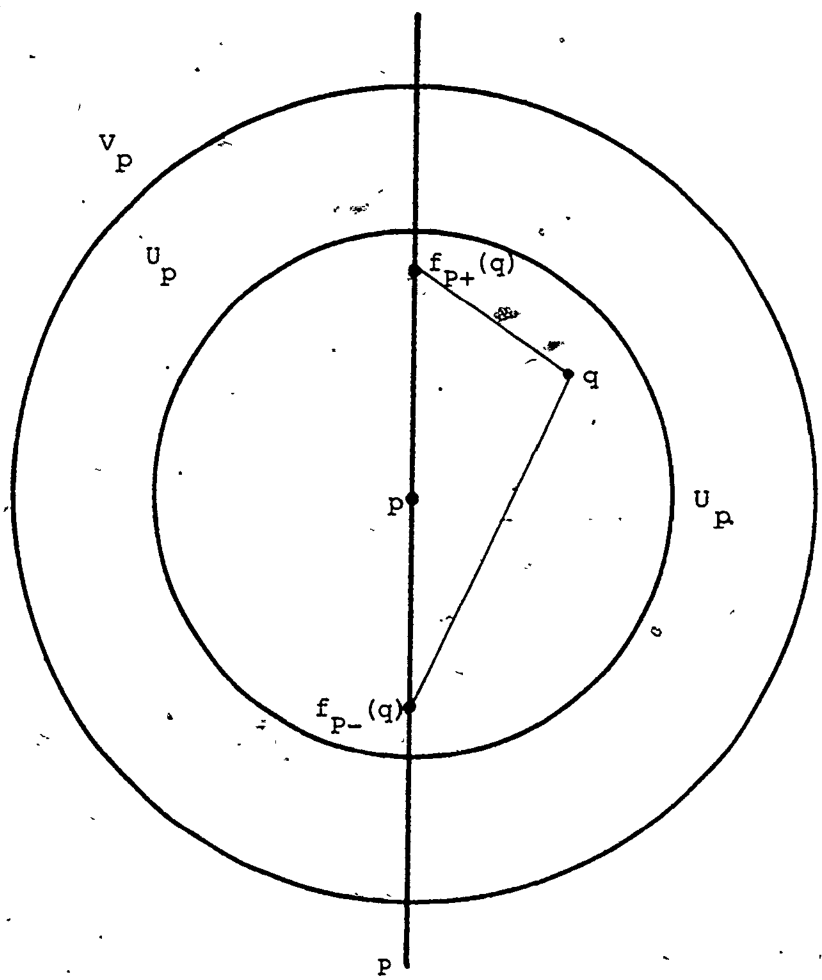
$\{L(p), p \varepsilon \mathrm{Dom}\ f\}$ is such, that the initial 1-directions depend smoothly on p.

CONFORMAL STRUCTURE

Axiom 9: For all $p \varepsilon M$, let $U_p \subset V_p \subset M$ satisfy the condition of axiom 3, and let P be any particle through p, and let t_p be a coordinate on $P \cap V_p$. Then, the map $g: U_p \to P \cap V_p$, defined by (for $q \varepsilon U_p$)

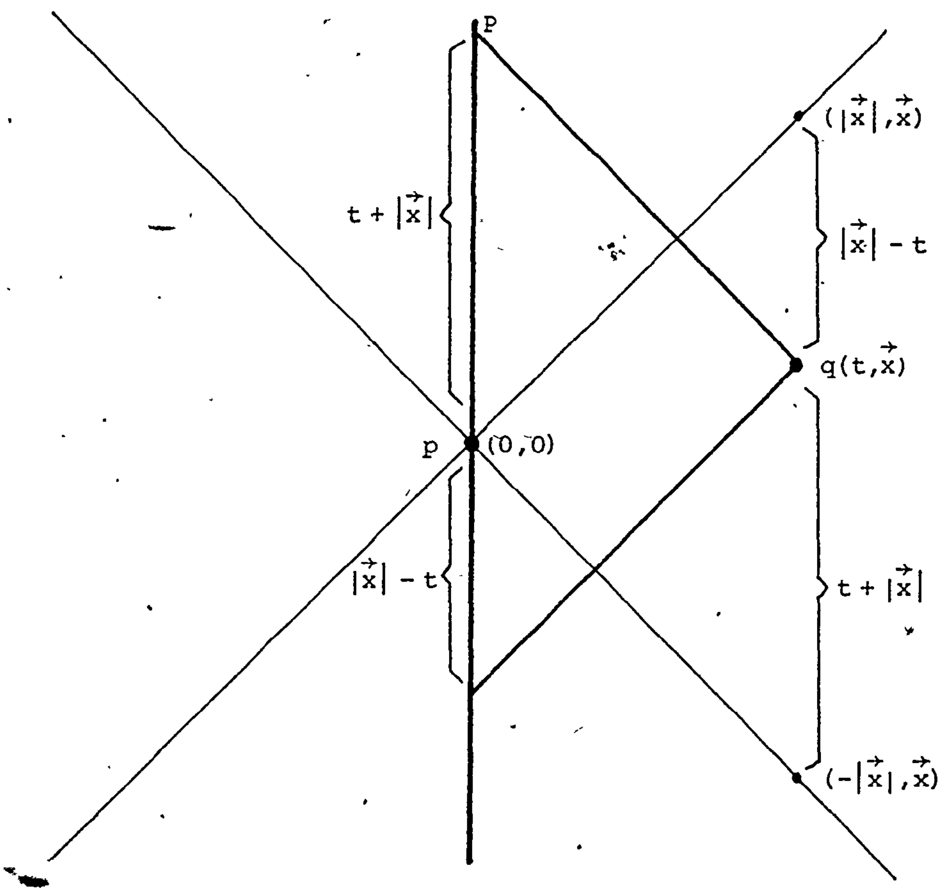
$$g(q) = -[t_p \circ f_{p-}(q) - t_p(p)][t_p \circ f_{p+}(q) - t_p(p)],$$

is C^∞. (C^5 would suffice)



Remark: In Minkowski spacetime, if the particle P is the time axis and the time coordinate is the standard one, then it is easy to show that g is the usual Minkowski metric.

Let p be an event at the origin, let q be an event at $(t,\vec{x})$ and let P be the particle the equation of which is $\vec{x}=0$.



$$
\begin{aligned}
g(q) &= (t+|\vec{x}|)(|\vec{x}|-t) \\
&= -(t+|\vec{x}|)(t-|\vec{x}|) \\
&= -(t^2-\vec{x}\cdot\vec{x}) \\
&= \vec{x}\cdot\vec{x}-t^2
\end{aligned}
$$

This gives the motivation for the definition of g.

Definition: For any event $p \varepsilon M$, the local light cone LLC_p at p is defined by

$$LCC_p = \{q \varepsilon U_p | f_{p-}(q) = p \text{ or } f_{p+}(q) = p\}.$$

It is clear that the elements of LLC_p can be separated into classes, each class lying on a light ray through $p \varepsilon M$. The next two axioms characterize the configurations generated by these light rays. The second axiom establishes the signature of the metric.

Axiom 10: For any $p \varepsilon M$, the set $\mathbb{LD}^1(M_p)$ of light directions at p, separates $\mathbb{D}^1(M_p) \setminus \mathbb{LD}^1(M_p)$ into two components.

Remark: The two classes of non-light directions correspond to timelike and spacelike directions. This axiom, however, does not guarantee past and future orientation; it allows the signature to be (−−++). The next axiom will force the signature of the metric to be ±2 and thus allows for the possibility to distinguish between future and past oriented vectors.

Axiom 11: Let $LT(M_p)$ denote the set of all elements of $T(M_p)$ which are nonvanishing and are tangent to light rays through p. Then $LT(M_p)$ consists of two connected components.

Theorem: There exists on M a smooth pseudo-Riemannian metric g_{ij} of signature (+++-), unique up to a smooth, positive scalar factor, such that at each event p, the vectors $X^i \varepsilon LT(M_p)$ tangent to light rays are characterized by $g_{ij}X^iX^j = 0$.

The theorem says that light propagation determines a conformal structure on M, that is, an equivalence class of conformally equivalent, locally defined metrics $[g] = \{\bar{g} | \bar{g} = e^{\theta} g\}$, where θ is any positive real valued function on M.

Geometrically speaking, a conformal structure defines everywhere a field of null cones on M. We can distinguish now between timelike, spacelike and null vectors and directions, and can define the orthogonality of vectors.

One may now proceed in one of two ways. One may work with some arbitrary representative of the equivalence class $\{e^{\theta} g\}$ and then verify that gauge-dependent terms, that is, terms which depend on the choice of the representative vanish, or one may choose a particular representative and then verify that the results are independent of this choice of gauge. EPS followed the second course, by choosing that representative which satisfies

$$\det(g_{ij}) = -1.$$

The g_{ij} denote the components of a tensor density of weight $-\frac{1}{2}$. One defines g^{ij} by

$$g^{ij}g_{jk} = \delta^i_k$$

and sets

$$K^i_{jk} = \tfrac{1}{2}g^{i\ell}(g_{\ell j,k} + g_{\ell k,j} - g_{jk,\ell}).$$

The equation for null geodesics then takes the form

$$\frac{dX^i}{dt} + K^i_{jk}X^jX^k = \lambda X^i$$

where $g_{ij}X^iX^j = 0$.

It is important to note, that the g_{ij} can be measured once a radar coordinate system has been defined. By sending nine light rays with linearly independent 'squared tangents' X^iX^j through p one may compute the g_{ij} by solving the system of linear equations

$$g_{ij}X^iX^j = 0, \quad \det(g_{ij}) = -1.$$

Moreover, as Ehlers pointed out, using more than nine light rays one can in principle check the axioms stated so far.

The next step is to show that light rays are conformal null geodesics. The g_{ij} were constructed pointwise and ensure only that light rays determine a conformal structure. That is, the null vectors are the tangent vectors of light rays. Hence light rays are conformal null curves.

<u>Theorem</u>: Light rays are conformal null geodesics.

PROJECTIVE STRUCTURE

We now turn to the discussion of the projective structure of spacetime. Within the context of the constructive axiomatic approach the problem arises of how to characterize the specific path structure defined by free (fall) motion. In other words, how is one to uniquely characterize the subclass P_f of P mentioned above. If a physical particle has internal structure, then in general its path in spacetime is not determined solely by its one-direction ξ_1 at some event. In general, the path will also depend on the orientation of the structure of the particle in spacetime. By their very definition spherically symmetric, massive, chargeless or charged particles do not have any internal structure which is sensitive to spacetime orientation. Consequently, the paths of such particles are determined solely by their 1-direction at a given event.

The main projective axiom in EPS constitutes an application of the infinitesimal law of inertia. The problem is therefore to be able to characterize free (fall) motion and avoid the circularity issues discussed in the introduction. What is required therefore is an analysis of general path structures which are characterizable in a way that assumes no more structure than is required for the possibility of constructing radar coordinates. For this reason, a general description of path structures was developed in [0] in terms of jet and direction bundles. A number of theorems were proven that serve as local differential topological criteria for singling out from the general path structures the geodesic ones that represent the inertial or the projective structure of physical spacetime. Moreover, these criteria can be employed within the constructive axiomatic approach as soon as radar coordinates (or any other coordinate diffeomorphic to the latter) have been introduced. That is, the criteria for geodecisity can be employed in the constructive axiomatic approach as soon as the local differential topological structure is introduced, the minimum structure required for setting up local radar coordinates. We shall elaborate on the philosophical implications of these remarks in the next chapter. For now we assume the projective axiom:

Axiom 12: There exists on M a unique projective structure Π (geodesic directing field) which determines the possible or actual paths of freely falling particles (members of P_f).

Remark: We wish to emphasize once more that this axiom is testable at a level that assumes no more structure than is required for introducing radar coordinates, namely, the local differentiable topological structure.

WEYL STRUCTURE

All that has been established so far is that there are actual or possible projective geodesics in every direction which is timelike as determined by the conformal structure. Very little connection between these two structures has so far been shown to exist. From a physical basis a close relation between the conformal and projective structure is suggested by high energy experiments: "A massive particle ($m > 0$), though always slower than a photon, can be made to chase a photon arbitrarily closely."[10] We therefore assume the axiom of compatibility.

Axiom 13: Each event p has a neighborhood U_p such that $q \varepsilon U_p$ and $q \neq p$ lies on a freely falling particle P through p if and only if q is contained in the timelike interior of the light cone.

One may deduce two main consequences from the compatibility axiom and the preceding results.

Theorem: A projective geodesic which is timelike, spacelike or null at one of its events, has the same causal character everywhere.

<u>Theorem</u>: All the null geodesics determined by the conformal structure belong to the class of geodesics determined by the projective structure. (Compatibility).

No affine parameters have been introduced so far and no process of parallel transport of vectors has been defined. On the basis of the preceding results we deduce a unique affine structure with the next theorem.

<u>Theorem</u>: The equivalence class Π of projectively equivalent symmetric linear connections contain one and only one member Γ^{i}_{jk} whose parallel transport preserves the conformal nullity of vectors.

Thus light propagation and free (fall) motion define on M a conformal structure and a unique symmetric linear connection such that light rays are conformal/affine null geodesics, freely falling particles are time-like projective/affine geodesics, and parallel transport maps null vectors into null vectors.

A spacetime endowed with such a structure is a Weyl space. As we saw in Chapter 4, Section 7, Weyl introduced such a structure in his attempt to construct a unified field theory. In a Weyl space the conformal structure singles out a unique symmetric linear connection Γ^{i}_{jk} from the equivalence class Π of projectively equivalent symmetric linear connections which characterizes the compatible projective structure on M. The Weyl connection determines the parallel transport of vectors, preserving their timelike, null or spacelike property, and for any pair of non-null vectors it leaves invariant their ratio of lengths and the angle between them. As we saw in Chapter 4, Section 7, in a Weyl space length transfer is non-integrable.

PSEUDO-RIEMANNIAN STRUCTURE

A Weyl geometry reduces to a Riemannian geometry if and only if the above mentioned integrability is satisfied, that is, if under parallel transport the magnitude of a vector is path-independent. Another way of stating this is to say that a Weyl geometry reduces to a Riemannian structure if and only if there is no second clock effect.

Since in a Weyl space parallel transport preserves the conformal nullity of vectors, the holonomy group consists of the product of a dilatation and a homogeneous Lorentz transformation. (The holonomy group of M at an event $p \varepsilon M$ is the group of all linear transformations of the tangent space $T(M_p)$ obtained by parallel displacement along closed curves starting at p.) From infinitesimal holonomic transformations, that is, parallel displacements along infinitesimal loops, it follows that the curvature tensor of the Weyl connection, namely

$$R^i_{jk\ell} = 2(\Gamma^i_{j[\ell,k]} + \Gamma^i_{m[k}\Gamma^m_{\ell]j}$$

decomposes into the form

$$R^i_{jk\ell} = \hat{R}^i_{jk\ell} + \tfrac{1}{2}\delta^i_j F_{k\ell}$$

where

$$g_{m(i}R^m_{j)k\ell} = 0; \quad F_{(ij)} = 0.$$

$\hat{R}^i_{jk\ell}X^kY^\ell$ corresponds to an infinitesimal Lorentz transformation and $\tfrac{1}{2}\delta^i_j F_{k\ell}X^kY^\ell$ to an infinitesimal dilatation.

$\hat{R}^i_{jk\ell}$ denotes Weyl's directional curvature (Richtungskrümmung) and F_{ij} is Weyl's length curvature (Streckenkrümmung). The vanishing of F, which amounts to the elimination of the dilatations from the holonomy group -- that is, vectors displaced in parallel along different paths from an event $p\varepsilon M$, where they are congruent, to another event $p'\varepsilon M$, will be congruent (relative to the conformal structure) also at $p'\varepsilon M$ -- is necessary for the existence of a Riemannian structure.

FOOTNOTES

1. [1] and [2].
2. [5], Chapter 22; [7], [8], and [12].
3. [4].
4. [6].
5. [6], pp. 6-7.
6. [3], pp. 685-686.
7. [9] in [10], pp. 195-207.
8. [11], p. 19.
9. [2], p. 28.
10. [2], p. 31.

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CHAPTER 6

THE REFUTATION OF GEOMETRIC CONVENTIONALISM

6.1 INTRODUCTION

What philosophical bearing does the geodesic method have on the question of the conventional or non-conventional status of the geometric structures of spacetime? Can it be argued, that the construction of a unique affine and pseudo-Riemannian structure, from a few qualitative assumptions concerning the local incidence and differential topological properties of light propagation and free fall, constitutes a convention-free and -- in relevant respects -- theory-independent body of evidence that can adjudicate between spacetime geometries and hence between spacetime theories that postulate them?

Recent criticisms of EPS's Constructive Axiomatics all suggest that the causal-inertial method is not convention-free and that the geodesic hypotheses are in relevant respects theory-dependent assertions, thereby rendering the geodesic method ineffective in providing a possible solution to the controversy between geometrical realism and conventionalism in favour of realism.[1]

In particular, Grünbaum has made the strong claim that any criterion which determines which bodies are suitable as test bodies, that is, which bodies are in free (fall) motion, and permits their identification, not only presupposes the specification of geometric structures beyond those implied by the local differential topological structure of spacetime, but such a criterion would necessarily presuppose metrical considerations thus involving EPS's geodesic method with logical and derivatively with epistemological circularity. Particles with a gravitational multipole structure, or charged particles in the presence of an electromagnetic field, will not in general travel along timelike geodesics. The only way of knowing when no forces act on a body is that it moves as a free particle along the geodesics of spacetime. So without already knowing the spacetime structure how is one to know which particles are neutral gravitational monopoles and which are not?

It is clear that these circularity charges are directed against EPS because they critically employ the notion of free (fall) motion -- that is, the infinitesimal version of the Law of Inertia -- without providing an explicit non-circular criterion for what is to count as a <u>free particle</u>. To the extent that EPS fail to supply an independent characterization of free (fall) motion in a way that would render their approach non-circular, these criticisms are certainly justified. To the extent, however, that they exploit the longstanding problem surrounding the Laws of Inertia, namely, that these Laws do not supply independent criteria of what is to count as <u>free motion</u>, the circularity charges directed against EPS are symptomatic of a longstanding misunderstanding and confusion concerning the meaning and content of the Inertial Laws. This confusion and misunderstanding is due in part to how the laws have been traditionally formulated.

In particular their meaning and content is obscured if one stays within the space plus time formulation. In that context, understanding Newton's First Law poses severe problems with respect to the following questions: What reference or coordinate system should one use to describe the motion of a body? What system of temporal measurement should be used? What system of spatial metric should be adopted? How is one to judge the presence or absence of forces?

One has tried to define a free particle with respect to an inertial frame such that Newton's First Law is realized by a free particle satisfying the equation $d^2x^i/dt^2 = 0$ in that frame. But how is one to determine what an inertial frame is? If one characterizes an inertial frame as a frame which satisfies the equation $d^2x^i/dt^2 = 0$ that a free particle would satisfy in that frame, then such a definition is obviously circular. Consequently, one sometimes interprets Newton's First Law to be an existence claim concerning physical inertial frames in which a free particle would satisfy $d^2x^i/dt^2 = 0$. Of course, a space plus time formulation of Newtonian gravitation theory requires strictly speaking that the above be construed counter-factually: If there were inertial frames (i.e., if gravitational forces were absent), free particles would satisfy $d^2x^i/dt^2 = 0$ in them. From a covariant four-dimensional point of view these difficulties disappear and the content of Newton's First Law constitutes a coordinate or frame independent empirical assertion about the projective structure of spacetime.

It is not surprising that in light of these difficulties which arise within the context of a space plus time formulation of Newton's First Law a multiplicity of claims concerning its status and meaning have been put forward:

(1) The Law of Inertia is synthetic a priori in character.

(2) The Law of Inertia does not constitute a proposition that expresses knowledge; whether that knowledge is hypothetical or melioristic in character. The Law is a general guiding principle for the acquisition of order of our empirical knowledge.

(3) The Law of Inertia is an empirical generalization which is obtained from observable phenomena on the basis of inductive procedures.

(4) The Law of Inertia is not an empirical hypothesis but constitutes a theoretical hypothesis that may be suggested by observable experimental data; but it cannot be justified by the latter. Observable phenomena may only "confirm" the law in at best an indirect way.

(5) The Law of Inertia has no empirical content. It is in truth a disguised definition or stipulation and conventional in character.[2]

We adopt exclusively the covariant four-dimensional point of view. All spacetime theories, whether classical or relativistic are covariantly formalizable. It is spacetime that constitutes the basic spatio-temporal entity of all spacetime theories. Failure to construe talk about space and time as talk about spatial and temporal aspects of spacetime will inevitably lead to misunderstandings and confusions concerning the meaning and content of the Laws of Motion.[3]

The transition from the kinematic to the dynamic behavior of bodies in spacetime theories has traditionally been viewed to consist in singling out a particular class of motions as the standard or natural motions which are considered as the free motions. The concept of force is then defined in terms of accelerations relative to these standard free motions.

The above procedure is meaningful if and only if there exists one and only one standard motion through each event and timelike direction at that event. Geometrically speaking, the choice of a class of standard motions amounts to giving the spacetime manifold a path structure. Such a path structure is geodetic and constitutes a projective structure on the spacetime manifold.

The definability of forces requires the specification or postulation of some unique projective structure of spacetime which provides a unique standard of no-acceleration relative to which forces may then be defined. Since geometrical conventionalism takes physical spacetime to be factually definite only up to local differential topological description, the projective, conformal, affine and metrical structures are conventionally assignable features of space-time. According to geometric conventionalism, what counts as a standard of no-acceleration or uniform motion is not dictated by a physically real, causally efficacious dynamic structure of spacetime. Rather, forces are definable relative to a chosen standard of no-acceleration, or equivalently, forces are conventionally assignable relative to some conventionally assignable projective structure of space-time, depending ontologically on that conventional choice.

One introduces the concept of a class of standard free motions and hence a projective structure by postulating the Law of Inertia. Hence, in terms of the above discussion, conventionalism considers force descriptions in terms of the Laws of Motion wholly conventional, such descriptions being constrained only by what we find convenient and elegant to use.

In particular, it is clear that Grünbaum considers the Laws of Inertia (classical and infinitesimal version (free fall)) to be non-empirical statements. The laws are conventional in the sense that their metrical forms can be conventionally altered. So viewed, they are translated into statements about the definition of spatio-temporal 'congruence' or 'force'. It is possible on this view that there be 'acceleration' in the absence of appropriate 'forces', provided we change our definition of 'congruent'. What counts as acceleration and force depends ontologically on the conventional choice of the metric. Hence forces, construed as causally active dynamical agents, are fictional entities.

Brian Ellis has attempted to establish the conventional assignability of forces independently and over and above considerations of metrical conventionality by focussing on the circularity problem concerning the notion of a 'free particle'.[4] The First Law states that free particles follow affine geodesics. But what is a free particle? A free particle is an instance of a zero-force situation. But how do we mark off zero-force situations from non-zero ones independently of the First Law? For to consider the First Law as our only criterion which singles out those situations where no forces are present is to make the law into a definition of 'free particle'.

Central to Ellis' argument for the conventional assignability of forces is the claim that zero-force situations are conventionally assignable. According to Ellis our only access to forces is through their effects, and those are nothing but deviations from some chosen standard of natural motion. Hence forces are not real members of our physical ontology, for if they were, then those situations in which they would assume a zero-magnitude could not be conventionally chosen ones. Forces exist only, Ellis tells us, "because we choose to regard some succession of states as an unnatural succession".[5] Consequently, the First Law of mechanics constitutes a definition of Newtonian force and may therefore be replaced by an alternative definition.

One of my main objectives is to provide a new formulation of the Law of Inertia which explicitly shows that it is a coordinate or frame independent empirical assertion about the projective structure of spacetime. Moreover, I shall show that the Law of Inertia is testable in a way that assumes no more structure than the local differential topology of spacetime. That is, it is testable at a level at which one need assume no more structure than is necessary for introducing arbitrary local coordinates. I will further show how this answers the question as to what extent EPS's geodesic method in deriving a unique pseudo-Riemannian metric constitutes a solution toward the controversy between realism and geometrical conventionalism in favour of geometrical realism. Since this analysis establishes that we do have epistemic access to light propagation and free (fall) motion (and hence to the conformal and projective structure of spacetime respectively) in a way that does not beset the geodesic method with either logical or derivatively epistemological circularity, I claim to have shown that EPS's approach decisively undercuts geometrical conventionalism. In fact I shall argue that the Constructive Axiomatics may be interpreted as constituting -- in relevant respects -- a theory-independent body of evidence that can adjudicate between spacetime geometries and hence between spacetime theories that postulate them.

In Section 2 the relevant philosophical concerns are presented that bear on the question whether the causal-inertial method for discovering the metric structure of spacetime provides a possible solution to the controversy between geometrical realism and conventionalism in favour of realism.

In Section 3 I show that the probative systems required for the measurements of the causal and inertial fields can be characterized purely in terms of the local differential topology of spacetime. Moreover, I show that there exist purely differential topological criteria which may be employed as empirical criteria to decide whether the inertial field is or is not geodesic. After noting that these mathematical criteria can only be used as empirical criteria provided that the structural fields can be measured, I underscore the importance of the existence of a direct measurement procedure for directing fields by pointing out the additional difficulties that prevent a direct measurement of the affine structure of spacetime.

Section 3 ends with a formulation of the Law of Inertia as an existence claim. The Law of Inertia says that there exists a unique projective structure on spacetime. I then define Free Motion and state the Law of Motion in terms of this unique projective structure and argue for the empirical status of the Law of Inertia and for the status of the projective structure and of forces (viewed as causally active dynamical agents) as physically real entities.

6.2 THE PHILOSOPHICAL SIGNIFICANCE OF THE CAUSAL-INERTIAL METHOD

The recent criticisms of EPS's Constructive Axiomatics all suggest that the causal-inertial method is not convention-free and that the geodesic hypotheses are in relevant respects theory-dependent assertions, thereby rendering the geodesic method ineffective in providing a solution to the controversy between geometrical realism and conventionalism in favour of realism.

I shall briefly discuss some of these criticisms by focussing on their overall virtue and strength in divesting the causal-inertial method of any ontological significance that might threaten conventionalism.

All of the charges laid against EPS concentrate on the roles which particles and to a lesser extent light rays play in EPS's formalism. We begin with Sklar who says:

> When we move on to the role particles play in the EPS formalism, we see that any hope of totally undercutting "conventionality" theses about the metric is unsupported by this formalization. The EPS formalism assumes that we know what a free particle is. And I suppose we do, given our vast array of background theory. But to support a "nonconventionality" thesis we would need reason to believe that we could determine when a particle was free totally independently of our theoretical assumptions. But how could we do this? Certainly not by seeing that it followed timelike geodesics! And if a particle is claimed to be free, we can always "conventionally" deny this by postulating "universal forces". After all, is a particle free or not when it is gravitationally attracted by another particle? If we say it is, and use the EPS construction, we shall get one spacetime. But what if we say it is not? Won't we

> get a whole "conventionally alternative" spacetime by using the EPS construction?
>
> . . .
>
> Now the choice of free particles to pick out the timelike geodesics in the EPS formalism follows because, according to general relativity, free particles travel timelike geodesics. But not all free particles do -- only those that are spherically symmetric and spin-free. For if the particle has a gravitational multipole structure, it will not generally travel the timelike geodesics, even if no forces act on it. And how, without already knowing the spacetime structure, are we to determine which particles are gravitational monopoles and which are not?[6]

Grünbaum's criticism of the geodesic method falls into two parts. He first argues that the criteria that determine which bodies are suitable as freely falling test bodies and how they could be recognized as such, presuppose metrical considerations, thus involving the geodesic method in logical and derivatively, epistemological circularity. He then argues that the path structure of spacetime singled out via light propagation and freely falling particles are not intrinsic properties of spacetime that can be discovered but are stipulated by us. We shall begin with the issue of logical and epistemological circularity.

Logical circularity: The selection of a suitable physical system by means of which Weyl and EPS assume we can discover the conformal and projective structure and ultimately a unique pseudo-Riemannian metric, necessarily presupposes measurements, and hence necessarily involves metrical assumptions. EPS's geodesic method, Grünbaum says

> must be able to avail itself, in a logically non-circular way, of the distinction between gravitationally monopole and multipole free massive test particles. For it must exclude the ST -- trajectories of the latter when stipulating that only the worldlines of the former are to be coextensive with the timelike geodesics of the desired Riemannian metric. And of course the latter metric is first going to be made available logically in the geodesic method by combining this geodecisity stipulation with the conformal structure of light rays. Hence no part of the GTR which is predicated on the resulting metric may be presupposed, on pain of vicious logical circularity, in order to distinguish at the outset between the two species of free test particles when imposing the geodecisity requirement on only one of them.[7]

Epistemological circularity: How would one recognize and uniquely identify relativistically freely falling particles without presupposing the metric of the theory for which they are to count as freely falling particles? Grünbaum says:

> Consequent upon this logical circularity there is also the epistemological one of knowing how to identify without (tacit) appeal to the resulting metrical theory, the gravitationally right kind of free particle, when first trying to ascertain the metric by the geodesic method. The geodesic method could become vitiated by patent epistemic circularity, if it were to seek to identify gravitationally monopole-free particles by first attempting to ascertain whether their ST -- trajectories are, in fact, geodesics.[8]

The charges of logical and epistemological circularity lead to the next level of argumentation designed to divest the geodesic method of any ontological significance that might threaten Grünbaum's thesis of ontological conventionality. Grünbaum's reasoning may be laid out roughly as follows:

P1: The four-dimensional spacetime manifold is metrically amorphous.

P2: There exist no intrinsic post-topological structures (e.g. metrical structures) that would allow us to single out in a non-conventional way a particular class of spacetime trajectories that have the properties of

i) time-like geodesicity

ii) metrical nullity

P3: The respective physical behaviors of photons (light rays) and of relativistically freely falling massive particles are pre-geometrical in character.

P4: Pre-geometrical facts or behaviors depend for their quantitative geometrical determination on the choice of a metric. (I.e. the choice or stipulation of a metric is ontologically constitutive of the geometrical determination of pre-geometrical facts or behavior).

P5: The selection of a class of spacetime trajectories possessing the geometrical property of being a

i) time-like geodesic

ii) null geodesic

depends on the stipulation of an indefinite

Riemannian metric. Thus the conformal and projective structures of spacetime are conventional.

P6: Compatibility of the conformal and projective structures of spacetime lead to a Weyl space with a unique affine structure. But since the conformal and projective structures are conventional, different pairs of compatible conformal and projective structures constitute different Weyl spaces.

P7: Given a Weyl space and hence a unique affine structure, a unique pseudo-Riemannian metric tensor can be deduced. But this metric is conventional because the Weyl space is.

The above criticisms exploit the fact that any description of the behavior of probative physical systems in spacetime that does not exclusively employ the local differential topology must use additional geometric structures. Since such structures are considered to be conventional different geometric assumptions can be made that will lead to different descriptions. Thus any method for finding the geometry of spacetime will be unsuccessful, if the corresponding probative physical systems used in the discovery procedure require for their description geometric assumptions beyond the local differential topology of spacetime. Therefore the causal-inertial method will of necessity be circular-if it is the case that one requires for the characterization of the behavior of light rays and freely falling particles those very structures, namely the conformal and projective structures, which are supposed to be revealed through their behavior. It is clear, that a genuine sense of "discovery" of these structures, amounts to being able to describe the behavior of probative physical systems revealing these structures, exclusively in terms of the local differential topological structure and to be able to distinguish between suitable and unsuitable test objects on that basis as well. Moreover, it must be possible to determine at that level whether or not these structures are unique.

Therefore any hope that the geodesic method can totally undercut geometrical conventionalism by providing a factual basis on which an empirical claim for the physical reality of the conformal and projective structures and hence metric structure can be made in a non-circular and non-conventional manner, requires that the descriptions of the probative systems used to reveal the conformal and projective structures of spacetime assume only the local differential topological structure.

But suppose we live in a devious world and are related to it in a way that makes it impossible for us to use merely the local differential topological structure for the description of any probative physical systems and that compels us to involve geometrical assumptions beyond those implied by the local differential topological structure. What would follow? In particular what would follow from the above criticisms which seem to suggest that something like this might be the case?

All that would follow is that we would then in fact be unable to identify or single out a class of suitable test objects which would have epistemic bearing on the spacetime geometry in an epistemologically non-circular way.

In particular, our inability to identify or single out a class of suitable test objects in an epistemologically non-circular way whose free motions would exhibit the unique projective structure of spacetime, if spacetime did in fact possess such a unique structure, means only that the truth of the Projective Axiom concerning free (fall) motion is epistemically undecidable. But any argument from the epistemic inaccessibility of free test particles does not establish that the structures derived from the axioms are ontologically conventional. A property which is objective and factual can at most be conventional in an epistemic but not in an ontological sense. Therefore we cannot know whether the projective structure of spacetime is ontologically conventional unless we first establish whether or not it is an objective feature of spacetime. Therefore, the most that is entailed by epistemological circularity is epistemological conventionality.

However, epistemological conventionality permits the assertion of the truth of the Constructive Axioms and hence the inference from them to a unique pseudo-Riemannian structure at least in this conditional sense:

> If the geometry-free axioms are true of the world and are hence satisfied by an actual or possible nonempty class of suitable test objects (light rays and symmetric nonrotating neutral, freely falling particles), then there exists a unique pseudo-Riemannian metric.

The truth of this conditional claim is incompatible with the truth of ontological conventionalism, for if the latter were true, then there could be no factual reasons, known or unknown, for preferring one metric over another. But EPS have at least shown that certain facts, if true and if known, would allow us to single out a unique spacetime metric. That we may not perhaps avail ourselves of these facts in an epistemically non-circular way supports only epistemological conventionalism.

Now it is one thing to show that the arguments put forward by conventionalism are not strong enough to establish the ontological conventionality of the geometric structures of spacetime. It is quite another thing to show that the conclusions they do draw are false in our particular world. An effective reply from the realist must therefore be a two-edged one.

On the one hand, the realist must show that the conventionalist's arguments are too weak to sustain the conclusions that he wants.

On the other hand, for the realist to make his case successfully, it is not sufficient to show that the arguments for the conventional assignability of the projective, conformal, affine and metric structures are inadequate, but he must actually succeed in showing the conventionalist's conclusion to be false in our particular world by showing that these geometric structures are real. In particular, he must succeed in showing that the world does not lack a physically real projective structure, that the transition from kinematics to dynamics is non-arbitrary but is in fact dictated by a physically real, causally efficacious unique projective structure.

Since the realist cannot argue for the physical existence of these geometrical structures on a priori grounds, something of an empirical claim has to be made. But in order to incisively undercut geometric conventionalism, the factual basis on which such an empirical claim for the physical reality of the projective, conformal, affine, metrical structures and hence forces rests, must be factually definite in a way that makes it immune to the charge that it, too, is conventional. Since the conventionalist views the world to be factually definite and non-conventional only up to local differentiable topological descriptions, it is clear that any decisive challenge to conventionalism requires some demonstration that the projective, conformal (causal), affine, metric structures and hence forces are physically real entities by assuming only the local differential topological structure.

In Section 3 I will show that the causal-inertial method, when supplemented with a number of theorems that serve as local differential topological criteria for singling out free (fall) motion at a level of testing that requires no more structure than is needed for introducing <u>arbitrary</u> local coordinates, is not beset with either logical or epistemological circularity. The geodesic method so supplemented constitutes a decisive challenge to conventionalism in that it provides a non-conventional factual basis for establishing the physical reality of the projective, conformal, affine and metric structures of spacetime.

6.3 DISCOVERING THE PROJECTIVE STRUCTURE[9]

It is shown in the Appendix that there exist purely topological criteria for deciding whether acceleration fields and directing fields are or are not geodesic fields (Theorems A4.1, A6.1, A6.2).

Moreover, it is shown that there is a one to one correspondence between geodesic acceleration fields and affine structures and between geodesic directing fields and projective structures.

The statement that the mathematical criteria for geodecisity are formulated exclusively in terms of local differential topological concepts must not be understood as having the force of saying that the existence of the property of geodecisity of either the acceleration field or directing field (and hence the existence of the affine and projective structure) is definable in terms of the local differential topological structure. Rather, if M is a differential manifold, then the necessary and sufficient conditions characterizing affine and projective structures on M -- if M should admit or possess such structures -- are expressible in terms of purely local differential topological concepts.

This is completely analogous to the way structures are defined in general. To use a simple example, let S be an object with the structure of a set. Then S may or may not possess a topological structure. But the necessary and sufficient conditions characterizing a topological structues on S -- should S possess or admit such a structure -- are expressible exclusively in terms of set theoretic concepts.

Since the mathematical criteria for geodecisity involve local differential concepts only,they can be effectively applied as <u>empirical</u> criteria,only if it is possible to measure the acceleration and directing fields by making use of only the local differential topological structures of spacetime. This amounts to the requirement that there exists a non-empty set of probative systems which are governed exclusively by these fields and whose behavior is characterizable purely in terms of the local differential topological structure required to set up local coordinate systems or radar coordinates.

If only the local differential topology is presupposed in the measuring procedure then acceleration fields cannot, in contrast to directing fields, be measured directly. To measure an acceleration field

$$A: H_1^1(M) \to H_1^2(M),$$

given in terms of local coordinates by

$$A(x^i(p),\gamma_1^i) = (x^i(p),\gamma_1^i,A_2^i(x^i(p),\gamma_1^i),$$

it is necessary to determine a large number of <u>solution curves</u> γ. This means, however, that in addition to their image paths $\gamma_{\vdash}(\mathbb{R})$, which can be determined directly, the parametrization must also be measured. It is the parametrization which cannot be measured without recourse to an indirect procedure. The parametrization is an image of the field structure of $\mathbb{R}$ which is <u>extrinsic</u> to the differential topology of M.

In the special case of the affine structure of spacetime (geodesic acceleration field) there does however exist an indirect procedure for measuring the corresponding acceleration field that uses only the local differential topological structure of spacetime. The indirect procedure makes use of the projective and conformal structure and a compatibility condition between these structures. First consider the projective structure. It will be seen shortly, that it can be measured directly using only the local differential topology. It defines on spacetime an equivalence class of projectively equivalent affine connections. The projective structure on M is a cross section Π: $M \to H^2(M)/P^2_n$ of the associated fiber bundle $H^2(M)/P^2_n$ of equivalence classes. For each $p \varepsilon M$, $\Pi(p)$ is an equivalence class of P^2_n-related n^2-frames, where P^2_n is the projective group. (See Section A7) Secondly, consider the causal structure of spacetime which corresponds to a unique first order conformal structure, namely, a field of infinitesimal light cones. The first order conformal structure of spacetime can be measured using only the local differential topological structure of spacetime. By a purely mathematical process involving only differentiation, the first order conformal structure determines a second order conformal structure. The latter defines an equivalence class K of conformally equivalent affine connections. If the projective and conformal structures are compatible then the intersection of the equivalence class Π (of projectively equivalent affine connections) and the equivalence class K (of conformally equivalent affine connections) contains a <u>unique</u> affine connection (Weyl connection). Consequently, although the method of measuring the affine structure (<u>geodesic</u> acceleration field) of a Weyl spacetime, is <u>indirect</u>, since it involves two fields, namely, the causal and the geodesic directing field (guiding field), only the local differential topological structure is required to measure and to determine the Weyl geometry of spacetime. This indirect procedure which presupposes no more structure than is required for setting up local coordinates (within the context of EPS, radar coordinates) is possible essentially because the parameter transformations of curve and surface elements are not compatible except in first order. Note that a light cone is a well behaved submanifold of M in the neighborhood of any of its points except the vertex. Furthermore, a surface is defined by a map $f: M \to \mathbb{R}$ whereas a curve is defined by a map $\gamma: \mathbb{R} \to M$. It is this assymmetry which accounts for the difference in the character of the parameter transformations beyond first order.

Let us now characterize more precisely the role of the differential topological criteria for geodecisity as empirical criteria within the context of the causal-inertial method for discovering the metric structure of spacetime.

C1: Suppose it is possible to establish a system of radar coordinates in the region of spacetime under investigation.

C2: Suppose it is possible to track material bodies with respect to this radar coordinate system. That is, the world line paths of material bodies can be determined by making use of only the local differential topological structure of spacetime required for setting up local coordinates, that is, radar coordinates.

C3: Suppose further that there exists a non-empty set of probative systems the motions of which are governed exclusively by a directing field.

Then the following can be shown to be the case:

C4: Given C1, C2, C3, the directing field can be measured directly (in the sense that no other fields are required) by using no more structure than is needed for introducing radar coordinates.

C5: Once the directing field is measured the differential topological criterion for geodecisity can be applied to determine whether or not it is also geodesic.

Hence if there exists a set of particles whose motion is exclusively governed by a given directing field, then it is possible to isolate those particles uniquely on the basis of C1 and C2 and hence on the basis of the differential topology. The set of particles thus singled out in this manner constitutes a specific <u>directing field set</u> of objects which are exclusively governed by that specific directing field. The directing field may now be measured by tracking a sufficiently large number of particles belonging to this specific directing field set. Once the directing field is measured the criterion for geodecisity may be applied to see whether it is also a geodesic directing field (guiding field). Therefore, if spacetime has a unique projective structure and if there exist particles which are exclusively governed by that structure, then we are able to determine that structure empirically at that level of empirical analysis which assumes no more structure then is required for setting up radar coordinates, namely, the local differential topology of spacetime.

It should be remarked here that the condition that there exist a class of particles which are exclusively governed by a geodesic directing field in order to measure that field and to determine its geodecisity by means of them can be considerably weakened. It is shown in [3] that even if particles governed solely by the projective structure of spacetime do not exist it is still possible to measure the projective structure of spacetime using more general probative systems, namely, charged monopoles. It is also shown there that charged monopoles may be used to measure not only the projective structure of spacetime but also the ratios of the components of the electromagnetic field as well as the conformal structure of spacetime.

We shall now show more precisely how directing fields may be measured (C4) and how they may be characterized as geodesic or non-geodesic (C5) provided that the three minimal conditions C1, C2, and C3 are satisfied.

One possible way of setting up a system of coordinates using light rays and material particles which may move along arbitrary (timelike) paths has been discussed in Section 5.4. It should not be supposed, however, that the use of radar coordinate systems is the only way to satisfy conditions C1 and C2. Whatever system of local coordinates is adopted, if several similar systems are used to chart the same phenomena, then for each pair of such systems, there must exist a smooth invertible transition map which translates the description of one system into that of the other, and for three such systems the transition maps must satisfy the appropriate transitivity relation.

Since the measurement procedure of directing fields is independent of the coordinate system used, one is free to make some convenient choice. One may even be guided in one's choice by specific theoretical assumptions and entertain, moreover, certain preconceptions concerning the relationship between the directing field to be measured and the chosen coordinate system. Since the measurement procedure is coordinate independent, one will be able to discover whether or not these preconceptions are justified.

The spacetime paths of particles whose motion is governed exclusively by a directing field is characterized by the fact that under arbitrary physical conditions, their trajectory is uniquely determined by an event on the trajectory and their direction ξ_1^α (three velocity, see equation A5.4) at that event. Particles that are exclusively governed by the same directing field can be made to fly in formation under arbitrary physical conditions, whereas structured particles may sometimes fly in formation but not at other times depending on the physical conditions.

A type of particle which will not fly in formation under all physical circumstances is the electron. It is well known that if a converging or diverging magnetic field is present, then the subsequent path of an electron depends not only on the given spacetime event and 1-direction ξ_1^α

at that event but also on the orientation of its magnetic dipole moment. Of course, if non-uniform fields are not present, then the motion of electrons will be determined by a directing field. In the case of charged particles with spin, the effect of a non-directing field behavior is quite dramatic. Consequently, it is easy to distinguish charged particles with spin since there exists physical conditions in which they do not fly in formation. Unstructured particles, on the other hand, will always fly in formation, even in an inhomogeneous field, hence under arbitrary circumstances. It is this special character of the motions which are exclusively governed by a directing field, which permits the separation of particles into a family of sets. One set of this family of sets contains all particles that are not exclusively governed by a directing field; all the other sets are distinct directing field sets each corresponding to a distinct directing field in the sense that its members are exclusively governed by that directing field. By definition all particles governed solely by a directing field are called monopoles, while all others are said to have a multipole structure.

Two particles belong to the same directing field set, if and only if whenever they are launched from infinitesimally near spacetime events with 1-directions ξ_1^α (three velocity) which differ only infinitesimally, their subsequent spacetime trajectories remain infinitesimally near.

It should be noted, however, that the claim that directions differ only infinitesimally at neighboring points does not presuppose a connection on the bundle $H^1(M)/D$ of projective n^1-frames of spacetime (see A5.7) since the infinitesimal projective transformations $G_n^1/D = PG(n-1)$ which relates the directions has been left arbitrary. Furthermore, the notion of 'near' in the above context, does not involve metric considerations. An appeal is made only to the local differential topological structure of spacetime. Of course the matching procedure is complicated by the fact that the local differential topological concept of nearness requires for its implementation limiting sequences of experiments. For our present concerns, however, it is only necessary that the matching procedures be possible in principle, that is, possible in theory, rather than simple or straightforward to implement.

Suppose a sufficient number of particles that are governed by a distinct directing field have been singled out. That is, a sufficiently large, distinct directing field set of monopole particles has been singled out by the above matching procedure which involves only local topological considerations.

We are now in a position to <u>measure</u> this directing field. As discussed in the Appendix, Section A6, a directing field

$$\Xi\colon \mathbb{D}^1_1(M) \to \mathbb{D}^2_1(M)$$

uniquely determines and is uniquely determined locally by the (2n-1) dimensional submanifold $\Xi_{\vdash}(\mathbb{D}^1_1(U))$ of the (3n-2) dimensional manifold $\mathbb{D}^2_1(U)$; $U \subset M$. By a procedure involving only differentiation, one may compute the lifts $j^2\xi$ of the solution paths ξ. These lifts are paths in the (2n-1) dimensional submanifold $\Xi_{\vdash}(\mathbb{D}^1_1(U))$. With respect to any given coordinate system, the equation of motion of a particle governed exclusively by a distinct directing field Ξ, has the form (A6.3), namely,

$$\frac{d^2x^\alpha}{(dx^n)^2}(x^n) = \Xi^\alpha_2(x^n, x^\alpha(x^n), \frac{dx^\alpha}{dx^n}(x^n)) \qquad (3.1)$$

where the spacetime coordinates are (x^n, x^α) and the three functions $\{\Xi^\alpha_2;\ \alpha=1,2,3\}$ uniquely define the directing field. The problem is then to experimentally determine three functions of (2n-1) variables. Stated in geometrical terms, it is necessary to determine in the space $\mathbb{R}^{10}$, with coordinates $(x^n, x^\alpha(x^n), dx^\alpha/dx^n, d^2x^\alpha/(dx^n)^2)$ the (2n-1) dimensional submanifold or hypersurface defined by the three equations 3.1.

The directing field may now be measured in a given spacetime region as follows. Take a sufficiently large number N of these particles that have been classified (by the procedure outlined above) as members of the corresponding distinct directing field set. Arrange to launch them from many locations in the spatial region of interest for some initial time slice and with a variety of three velocities ξ_1^α in such a way that the points representing their initial positions and their velocities are reasonably distributed over the space $\mathbb{R}^6$ with coordinates $(x^\alpha, dx^\alpha/dx^n)$ or $(x^\alpha(x^n), \xi_1^\alpha)$.

Then track each of the particles with respect to the chosen coordinate system (radar coordinate) obtaining the description of their world lines in terms of 3N functions $\{x_i^\alpha(x^n);\ \alpha=1,2,3;\ i=1,2,\ldots,N\}$.

Next, differentiate these functions to obtain the equations of N paths each of which lies in the (2n-1) dimensional submanifold of $\mathbb{R}^{10}$ which characterizes the directing field; namely, the equations

$$x^n \to \left(x^n, x_i^\alpha(x^n), \frac{dx_i^\alpha}{dx^n}(x^n), \frac{d^2x_i^\alpha}{(dx^n)^2}(x^n)\right) \tag{3.2}$$

for i=1,2,...,N.

Finally apply standard techniques of numerical interpolation to determine the (2n-1) dimensional submanifold corresponding to Ξ_2^α and hence the directing field Ξ.

Note, that in contrast to the case of acceleration fields, the particular choice of the parameter for curves which represent the paths is of no significance here and therefore does not affect the determination of the directing field. Moreover, it is important to re-emphasize that the measuring procedure for directing fields does not depend on the coordinate system that is used.

The next task is to determine whether or not the directing field that has been measured by the above procedure is geodesic or not. As is shown in the Appendix, Section 6, a directing field

$$\Xi: \mathbb{D}^1_1(M) \to \mathbb{D}^2_1(M),$$

where

$$\Xi(x^i(p), \xi^\alpha_1) = (x^i(p), \xi^\alpha_1, \Xi^\alpha_2(x^i(p), \xi^\alpha_1)),$$

is a geodesic directing field, if and only if relative to any given chart (U,x), the functional dependence of the directing field function $\Xi^\alpha_2(x^n, x^\alpha(x^n), dx^\alpha/dx^n)$ on the three velocities dx^α/dx^n is cubic at every spacetime event. Specifically, denote a geodesic directing field by Π and the three velocities dx^α/dx^n by ξ^α_1. Then a directing field Π is geodesic if and only if relative to any local system of coordinates $(x^i(p), \xi^\alpha_1, \xi^\alpha_2)$

$$\Pi^\alpha_2(x^i(p), \xi^\alpha_1) \tag{3.3}$$

$$= \xi^\alpha_1[\Pi^n_{\rho\sigma}(x^ip))\xi^\rho_1\xi^\sigma_1 + 2\Pi^n_{n\rho}(x^i(p))\xi^\rho_1 + \Pi^n_{nn}(x^i(p))]$$

$$-[\Pi^\alpha_{\rho\sigma}(x^i(p))\xi^\rho_1\xi^\sigma_1 + 2\Pi^\alpha_{n\rho}(x^i(p))\xi^\rho_1 + \Pi^\alpha_{nn}(x^i(p))],$$

where $\Pi^i_{ji}(x^i(p)) = 0$ so that $\Pi^n_{n\rho}(x^i(p))$ and $\Pi^n_{nn}(x^i(p))$ may be eliminated from (3.3).

It is apparent from (3.3) (Theorem A6.1) that if Π is a geodesic directing field, then $\Pi^\alpha_2(x^i(p),\xi^\alpha_1)$ is a cubic polynomial in ξ^α_1 in <u>every</u> coordinate chart $(U,x)_p$. The converse is also true (Theorem A6.2):

If with respect to every and hence any coordinate chart $(U,x)_p$ the corresponding map $\Xi^\alpha_2(x^i(p),\xi^\alpha_1)$, which determines the directing field Ξ, is cubic in the ξ^α_1's, that is, if

$$\Xi^\alpha_2(x^i(p),\xi^\alpha_1) = A^\alpha + B^\alpha_\rho\xi^\rho_1 + C^\alpha_{\rho\sigma}\xi^\rho\xi^\sigma_1 + D^\alpha_{\rho\sigma\tau}\xi^\rho_1\xi^\sigma_1\xi^\tau_1, \tag{3.4}$$

where the coefficients A, B, C, D are functions only of $p\varepsilon M$, then Ξ is geodesic.

Since the functions $\Xi^\alpha_2(x^i(p),\xi^\alpha_1)$ have been determined by the measurements described above, it is only necessary to evaluate the various partial derivatives of the functions $\Xi^\alpha_2(x^i(p),\xi^\alpha_1)$ with respect to the ξ^α_1 in order to check that (3.3) holds.

If the forth and higher order partial derivatives are not zero, then the directing field is not geodesic. Note that the projective coefficients $\Pi^i_{jk}(x^\ell(p))$ are just the appropriate partial derivatives of $\Pi^\alpha_2(x^i(p),\xi^\alpha_1)$. <u>These coefficients uniquely determine and are uniquely</u>

determined by a geodesic directing field.

Moreover, it is important to note the following: The assertion that a directing field Ξ is geodesic is falsified if it fails to have the functional form (3.3) at even a single spacetime event.

It is furthermore interesting to observe, that, given a coordinate system $(U,x)_p$, the term $\Pi^{\alpha}_{nn}(x^i(p))$ in (3.3) corresponds in the non-relativistic limit to the Newtonian part of the directing field, namely, the gravitational force, while the remaining terms correspond to, what has been called by analogy, "gravitational magnetism."

In summary, the identification of the class of particles governed by directing fields, the measurement of those fields and the criterion for deciding whether or not a given directing field is geodesic require reference only to the local differential topological structure of spacetime, namely, that amount of structure required to set up local coordinates (radar coordinates).

Unless it is realized that geodecisity is a differential topological invariant it may seem somewhat surprising that geodecisity may be empirically tested exclusively at the level of the local differential topology. Now the statement that geodecisity is a differential topological invariant does not mean that any given geodesic directing field is invariant under all diffeomorphisms.

Consider the set of all directing fields on a given manifold M and the subset of all geodesic directing fields. Then the differential topological invariance of geodecisity follows from the fact that the subset of geodesic directing fields remains stable under arbitrary diffeomorphisms even though the individual members of this subset are permuted among themselves. Hence the cubic polynomial form of the map $\Pi_2^{\alpha}(x^i(p),\xi_1^{\alpha})$ in the variables ξ_1^{α} is characteristic of the <u>class</u> of geodesic directing fields; that is it is characteristic of projective<u>ness</u>. That is an arbitrary diffeomorphism will transform a geodesic directing field into another directing field which is also geodesic.

The invariance of geodecisity under arbitrary diffeomorphisms is transparent when formulated in terms of the projective structures or P_n^2-structures on M. Indeed, it is just a special instance of the following general result:

Let H^k be the principal bundle of n^k-frames and let G_n^k be the corresponding structure group. Consider the diffeomorphism $\phi\colon M \to M$. Then a bundle automorphism (see Definition A1.1) is a pair of diffeomorphisms $\phi\colon M \to M$ and $j^k\phi\colon\ H^k(M) \to H^k(M)$, such that

$$\phi\circ\pi = \pi\circ j^k\phi. \qquad (3.5)$$

For all $p\varepsilon M$, the restriction of $j^k\phi$ maps fibers into fibers, that is,

$$j^k_p\phi\colon H^k(M_p) \to H^k(M_{\phi(p)})$$

and the following diagram is commutative:

$$\begin{tikzcd}
H^k(M) \arrow[rr, shift left, "j^k\phi"] \arrow[d, "\pi_{H^k}"'] & & H^k(M) \arrow[ll, shift left, "j^k\phi^{-1}"] \arrow[d, "\pi_{H^k}"] \\
M \arrow[rr, shift left, "\phi"] & & M \arrow[ll, shift left, "\phi^{-1}"]
\end{tikzcd} \tag{3.6}$$

If SG^k_n is a closed Lie subgroup of G^k_n then an SG^k_n-structure on a manifold M is a reduction of $H^k(M)$ to the subgroup SG^k_n of G^k_n (see Section A7). An SG^k_n structure on M is represented by a subbundle of $H^k(M)$, a reduced principal bundle with structure group SG^k_n. Such SG^k_n-structures are in bijective correspondence with cross sections of the associated fiber bundle $H^k(M)/SG^k_n$.

The bundle automorphism $j^k\phi\colon H^k(M) \to H^k(M)$ induces a bundle automorphism on the associated fiber bundle $H^k(M)/SG^k_m$. Then an SG^k_n-structure $\sigma\colon M \to H^k(M)/SG^k_n$ is transformed into an SG^k_n structures $\sigma^\phi\colon M \to H^k(M)/SG^k_n$ according to

$$\sigma^\phi = j^k\phi\circ\sigma\circ\phi^{-1}$$

and the following diagram is commutative:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
H^k(M)/SG^k_n \arrow[r, shift left, "j^k\phi"] \arrow[d, shift left, "\pi_H^k/SG^k_n"] & H^k(M)/SG^k_n \arrow[l, shift left, "j^k\phi^{-1}"] \arrow[d, shift left, "\pi_H^k/SG^k_n"'] \\
M \arrow[u, bend left, "\sigma"] \arrow[r, shift left, "\phi"] & M \arrow[l, shift left, "\phi^{-1}"] \arrow[u, bend right, "\sigma^\phi = j^k\phi\circ\sigma\circ\phi^{-1}"']
\end{tikzcd}
\qquad (3.7)
$$

Hence, if $SG^2_n = P^2_n$, then a projective structure $\sigma: M \to H^2(M)/P^2_n$ is transformed into a projective structure $\sigma^\phi: M \to H^2(M)/P^2_n$. The same applies to any other G-structure such as for example the affine structure ($SG^2_n = \Gamma^2_n$) or pseudo-Riemannian structure ($SG^1_n = O^1_{1,n-1}$).

The preceding analysis does not conflict with an earlier assertion which says that the projective structure uniquely determines and is uniquely determined by the projective coefficients. The reason is that directing fields are measured with respect to some chosen local coordinate system Σ. Therefore, any diffeomorphism $\phi: M \to M$ that transforms a directing field Ξ into another directing field Ξ^ϕ will also transform the chosen coordinate system Σ into another coordinate system Σ^ϕ (that is, the fields and world lines of physical systems used to construct Σ will be transformed into other fields and world lines) such that Ξ^{ϕ} will have the same relation to Σ^{ϕ} as Ξ has to Σ; that is, the map $\Xi_2^{\alpha}(x^i(p), \xi_1^{\alpha})$ which determines Ξ with respect to Σ also determines Ξ^{ϕ} with respect to Σ^{ϕ}.

It is not difficult to see that the above analysis has direct bearing on the question concerning the status of the Law of Inertia. The existence of a unique projective structure Π: $\mathbb{D}^1_1(M) \to \mathbb{D}^2_1(M)$ on spacetime determines a unique coordinate independent standard of no-acceleration. This geometric structural field bijectively corresponds to a unique geodesic directing field or differential equation for paths in spacetime. The world line solution paths of the field Π constitute a distinguished class of motions which are the free motions of material bodies with respect to which forces may be defined.

For <u>any</u> theory of spacetime in which the model of its event set is a four dimensional differential manifold M, the ontological basis of the dynamics of material bodies is expressed in the following <u>empirical</u> law:

THE LAW OF INERTIA: There exists on spacetime a unique projective structure Π (or equivalently, a unique geodesic directing field Π).

The concept of <u>free motion</u> may now be characterized on the basis of the Law of Inertia in the following way:

FREE MOTION: A possible or actual material body is in a <u>state of free motion</u> during any part of its history, just in case the corresponding segment of its world line path is a solution path of the differential equation determined by the unique <u>projective structure</u> of spacetime.

Newton's second law of motion may now be reformulated as follows:

THE LAW OF MOTION: With respect to any coordinate system, the path that describes the world line of a possible or actual material body satisfies an equation of the form

$$m(\xi_2^\alpha - \Pi_2^\alpha(x^i(p), \xi_1^\alpha)) = F^\alpha(x^i(p), \xi_1^\alpha), \qquad (3.8)$$

where m is a scalar constant characteristic of the material body, called its mass, and $F^\alpha(x^i(p), \xi_1^\alpha)$ is the force acting on the body.

The form of the equation (3.8) is independent of the coordinate system chosen provided that under a coordinate transformation, the forces transform according to

$$\bar{F}^\alpha(\bar{\xi}_1^\alpha) = \frac{\bar{\Lambda}_\rho^\alpha F^\rho(\xi_1^\alpha)(\bar{\Lambda}_n^n + \bar{\Lambda}_\beta^n \xi_1^\beta) - \bar{\Lambda}_\rho^n F^\rho(\xi_1^\alpha)(\bar{\Lambda}_n^\alpha + \bar{\Lambda}_\beta^\alpha \xi_1^\beta)}{(\bar{\Lambda}_n^n + \bar{\Lambda}_\sigma^n \xi_1^\sigma)^3}. \qquad (3.9)$$

This transformation law for forces follows from (3.8) and the transformation law (A5.6) together with a similar law for $\Pi_2^\alpha(x^i(p),\xi_1^\alpha)$ obtained by replacing $\bar{\xi}_2^\alpha$ and ξ_2^α by $\bar{\Pi}_2^\alpha$ and Π_2^α.

From the fact that the transformation law for forces is linear in the forces and does not contain an inhomogeneous term, it follows that forces are additive and that, if a force is nonzero in any coordinate system, then it is nonzero in all coordinate systems, for otherwise the projective structure of spacetime would not be unique.

The forces $F^\alpha(x^i(p),\xi_1^\alpha)$ may also depend on the variables which describe the orientation of possible internal multipole structure of the material body and the coupling of such structure to various geometric object fields in its microneighborhood. For those particles which have such additional internal structure, it is necessary to supplement (3.8) with additional equations of motion for the internal variables in order to obtain a complete system of equations adequate for <u>predicting</u> the motion of the particle. However, the Law of Motion (3.8) alone is adequate for the <u>analysis</u> of an observed motion and for the measurement of the force acting on a body, since the quantities appearing on the left side of (3.8) (ξ_2^α and $\Pi_2^\alpha(x^i(p),\xi_1^\alpha)$ can be empirically determined by ideal test procedures which presuppose only the different topological structure of spacetime.

It is important to emphasize that the Law of Motion (3.8) makes explicit use of the unique projective structure Π on spacetime. <u>The Law of Motion, therefore, depends ontologically on the Law of Inertia. Consequently, it is impossible to derive the Law of Inertia from the Law of Motion</u>.

The Law of Inertia and the Law of Motion as formulated above apply to <u>all</u>, relativistic or non-relativistic, curved or flat, dynamic or nondynamic, spacetime theories. The reason for the general character of these laws consists in the fact that they require for their formulation only the differential topological structure of spacetime, a structure which is <u>common</u> to all spacetime theories.

The Law of Inertia is an <u>empirical</u> law. It is <u>falsifiable</u>, for if there exist at least two classes of particles each of which is governed by a distinct geodesic directing field (projective structure), then as discussed above the particles belonging to these classes may be identified and used to measure in any chosen local neighborhood of spacetime the two distinct projective structures. The discovery in any local region of spacetime of two distinct projective structures is sufficient to falsify the Law of Inertia.

If there exists exactly one class of particles governed solely by a geodesic directing field, then the particles may be identified and used to measure the unique projective structure of spacetime in any chosen local neighborhood of spacetime by a procedure which is coordinate independent.

The fact that the motions of free particles are uniquely determined by the projective structure of spacetime makes them particularly suitable as instruments for probing this structure because their motions reveal the unique projective structure of spacetime in a particularly simple manner. However, as explicitly formulated above in the Law of Motion, the unique projective structure of spacetime constrains in part the motions of all material bodies. Hence, it should be expected that a sufficiently sophisticated analysis would permit the use of more complicated material bodies for the measurement of the projective structure of spacetime. A not too complicated example of such an analysis is provided in [3] for charged monopoles.

The measurability of the projective structure of spacetime at the local differential topological level constitutes an empirical nonconventional basis which establishes that the projective structure of spacetime is a real physical field.

It is clear that forces are measurable at a level that uses only the local differential topological structure of spacetime. For, only this structure is required to measure the projective structure Π, to measure the paths ξ of arbitrary material bodies and to compute the lifts $j^1\xi$ and $j^2\xi$. Aside from the inertial mass m which is just a constant, all quantities on the left side of the Law of Motion (3.8) are measurable. Consequently, the force acting on a material body is measurable. The measurability of forces just described establishes that forces, conceived of as causal dynamical agents, are real physical entities.

The main projective axiom in EPS constitutes an application of the infinitesimal version of the Law of Inertia. It is clear from the above, that this axiom can be tested at a level which makes use of only the local differential structure required to set up radar coordinates.

FOOTNOTES

1. See footnote (2) of Chapter 5.
2. [9], p. 114.
3. [4].
4. [6].
5. [6], p. 46.
6. [8], pp. 259-260.
7. [7], p. 747.
8. [7], p. 747.
9. This section is based on [1], [2] and [3].

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APPENDIX[1]

1. FIBER BUNDLES

A bundle is a triple $<E,\pi,M>$ consisting of two topological spaces E and M, and a continuous surjective map $\pi: E \to M$. E and M are called bundle space and base space respectively. The simplest example of a bundle is the cartesian bundle $<M_1 \times M_2, \pi_1, M_1>$ with π_1 the first projection defined by $\pi_1(p,q)=p$, for all $p \varepsilon M_1$ and for all $q \varepsilon M_2$. A simple concrete example of a cartesian bundle is a cylinder considered as a cartesian product of a one-sphere (circle) S^1 with a line segment I. That is, $<S^1 \times I, \pi_1, S^1>$.

If for all $p \varepsilon M$, the topological spaces $\pi^{-1}(p)$ are homeomorphic to some space F, then F is called the typical fiber and $\pi^{-1}(p)$ is called the fiber over $p \varepsilon M$ and is sometimes denoted by F_p.

If the bundle also has certain additional structures involving a group homomorphism of F onto itself, and a covering of M by open sets, then the bundle is called a fiber bundle.

A differentiable fiber bundle $\mathcal{E}(M)$ is a fiber bundle over a differentiable manifold M and its structure will be denoted by

$$\mathcal{E}(M) = <E(M), \pi_E, M, F, G> \qquad (A1.1)$$

1) The total or bundle space E(M), the base space M and the typical fiber F are differentiable manifolds.

The projection map π is surjective and differentiable.

2) For every $p\varepsilon M$, $\pi^{-1}(p)$ is called the fiber over $p\varepsilon M$. For every $p\varepsilon M$, $\pi^{-1}(p)$ is diffeomorphic to the typical fiber F, and F constitutes the <u>representation</u> or <u>coordinate space</u> for the fibers F_p.

For each type of geometric object, an appropriate fiber bundle may be constructed over the base manifold M. A geometric object is then a point in the bundle space (total space) E(M), that is, a point on some fiber F_p, $p\varepsilon M$. The point p of the base manifold M to which it belongs, is simply its base point.

A geometric object is <u>represented</u> by an element of some typical fiber space. This representation depends on which of the possible coordinate systems ($\psi_U|_p$, $\psi_{\bar{U}}|_p$, etc.) for the fiber at $p\varepsilon M$ is being used. The geometric object itself can be constructed only if both the representative (a point in the model space F) and the specific coordinate system $\psi_U|_p$ for the fiber at $p\varepsilon M$ are specified.

3) G is a Lie group, that is an r-dimensional differentiable manifold G with a group structure, such that the product map ϕ: $G\times G \to G$ and the inverse map μ: $G \to G$ are differentiable.

G acts on the typical fiber F. By convention this group action on F is from the left and this <u>left action</u> L: $G\times F \to F$ is differentiable and satisfies

$L(g_1 g_2, z) = L(g_1, L(g_2, z))$, $z\varepsilon F, g_1, g_2 \varepsilon G$.

The left group action is <u>effective</u>. This means that the only element of G which leaves every point of F fixed is the identity element $e\varepsilon G$. More precisely, a left action $L: G\times F \to F$ is <u>effective</u> if and only if

$$\forall g\varepsilon G[(\forall z\varepsilon F(L(g,z)=z) \Rightarrow (g=e)]. \tag{A1.2}$$

The elements of G are therefore distinguished by their action on F. That is, if two elements have the same action on F, then they must be identical.

4) The total space E(M) is <u>locally trivial</u>. For every coordinate neighborhood U of M there exists at least one differentiable map

$$\psi_U: \pi^{-1}(U) \to F \tag{A1.3}$$

such that the map

$$\pi\times\psi_U: \pi^{-1}(U) \to U\times F \tag{A1.4}$$

is a diffeomorphism. In other words, corresponding to each neighborhood $U \subset M$, the sub-bundle of E(M), namely

$$<\pi^{-1}(U), \pi_{\pi^{-1}(U)}, U, F, G> \tag{A1.5}$$

is isomorphic to the product bundle

$$<U\times F, pr_1, U, F, G> \tag{A1.6}$$

The restriction of ψ to some fiber over $p\varepsilon M$, namely

$$\psi_U'|_p: \pi^{-1}(p) \to F \tag{A1.7}$$

is a diffeomorphism. Moreover, if

$$\psi_{\overline{U}}:\pi^{-1}(\overline{U}) \to F$$

is another differential map such that $p\in U \cap \overline{U}$, then the diffeomorphism

$$\psi_{\overline{U}}|_p \circ \psi_U|_p^{-1}: F \to F \tag{A1.8}$$

coincides with the <u>action</u> of some element of G and therefore defines an element in G, since G acts effectively on F.

For all $p\in U_i \cap U_j \subset M$ and for all i,j of some index set, the diffeomorphisms

$$\psi_{U_j}|_p \circ \psi_{U_i}|_p^{-1}: F \to F$$

uniquely define an element of the <u>structural group</u> G of $E(M)$ and therefore induce the corresponding differential maps

$$g_{\psi_{U_j}\psi_{U_i}} \equiv g_{ji}: U_j \cap U_i \to G \tag{A1.9}$$

which are called the <u>transition functions</u> of the fiber bundle and are defined as follows: For all $p\in\overline{U} \cap U$ and for all $z\in F$,

$$L(g_{\overline{\psi}\psi}(p), z) = \psi_{\overline{U}}|_p \circ \psi_U|_p^{-1}(z). \tag{A1.10}$$

Some of the ideas are illustrated in the following diagram.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[d, "\pi"'] & & U \times F \arrow[dll, "pr_1"] \\
U & &
\end{tikzcd}$$

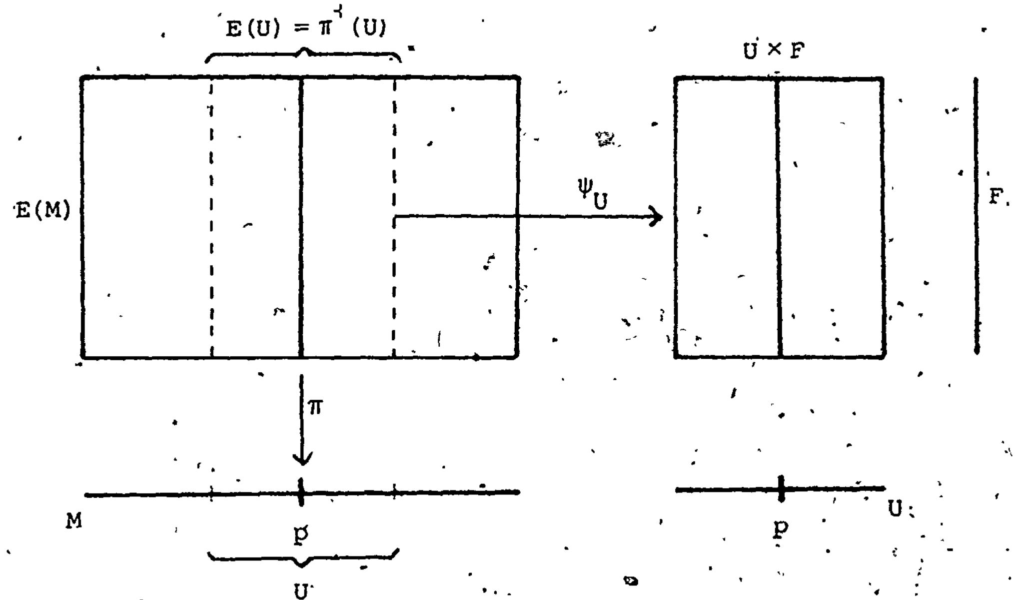


FIGURE A1

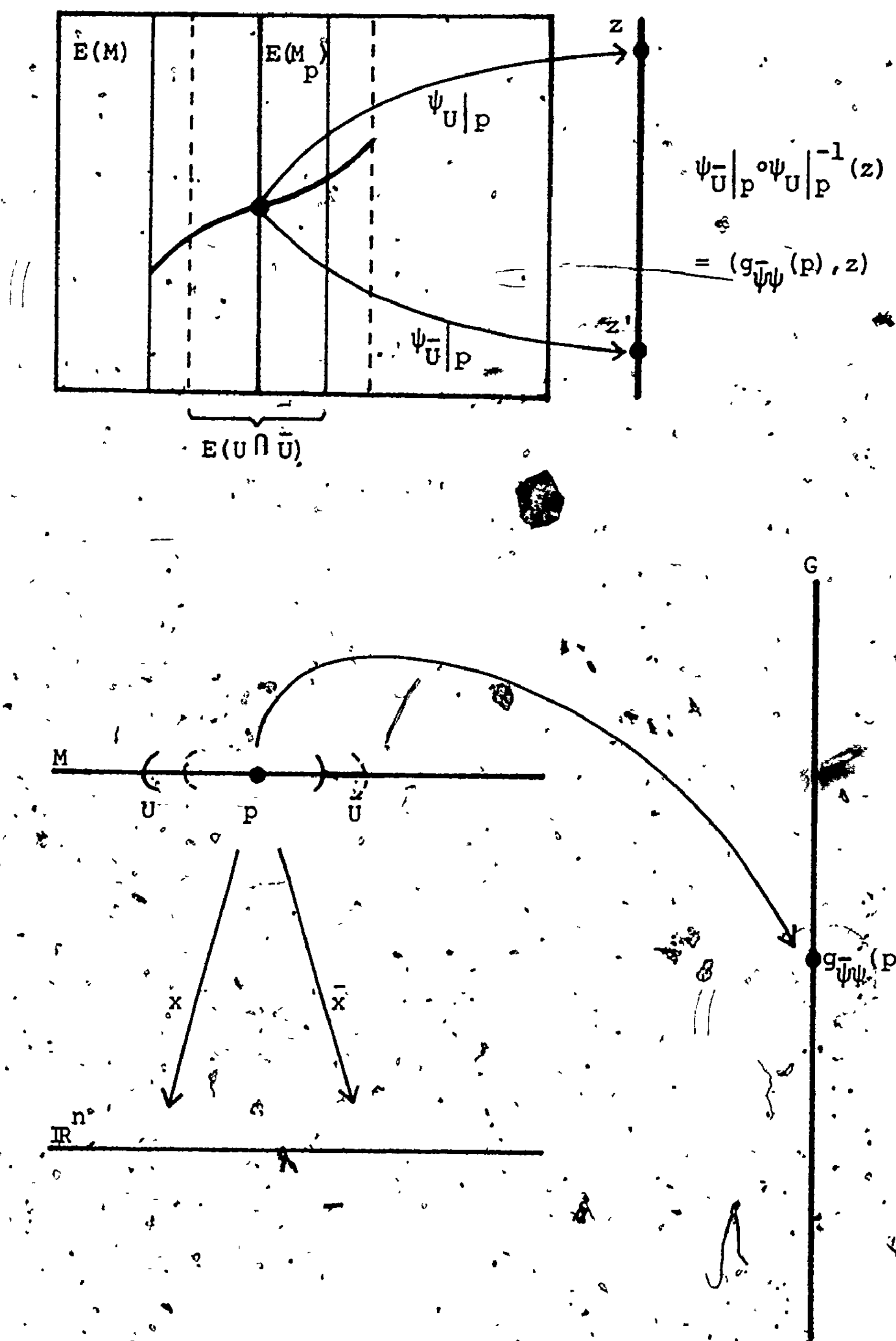


FIGURE A2

The representatives z and z' respectively (Figure A2) correspond to the charts $(U,x)_p$, and $(\overline{U},\overline{x})_p$. $(U,x)_p$ and $(\overline{U},\overline{x})_p$ induce the mappings $\psi_U|_p$ and $\psi_{\overline{U}}|_p$.

For many applications to spacetime structure the elements of the base manifold M represent the events of physical spacetime. The geometric objects of a given type on spacetime are represented as elements of the total space E(M) of the appropriate bundle E(M) over M. Furthermore, physical geometric object fields are represented by special maps called cross sections.

A cross section of a bundle E(M) is a map $\sigma: M \to E(M)$ which satisfies $\pi \circ \sigma = id_M$.

A local cross section E(M) is a cross section of the portion $E(U) = \pi^{-1}(U)$ over a coordinate neighborhood $U \subset M$.

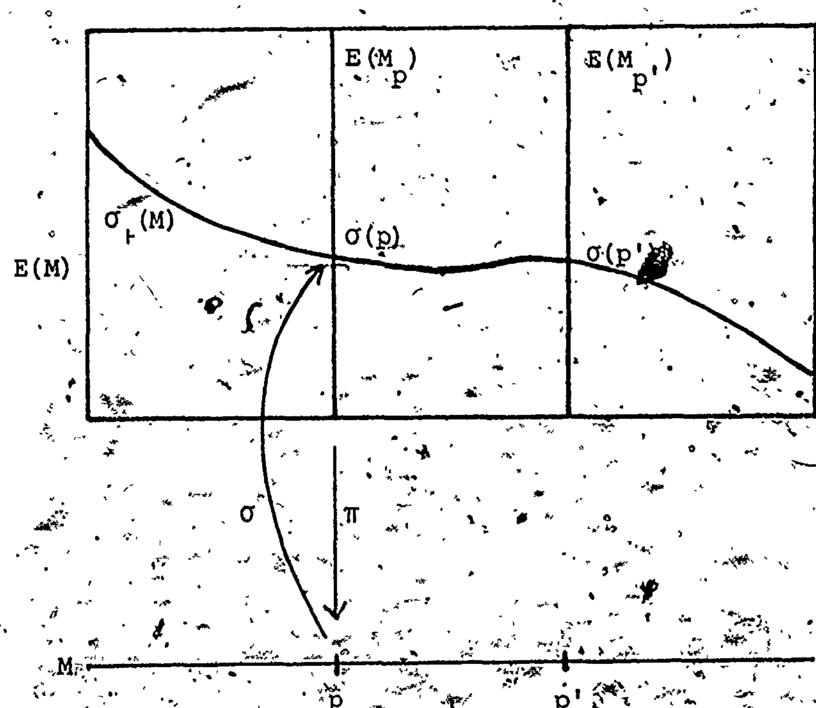


FIGURE A3

PRINCIPAL BUNDLE

A fiber bundle $\mathcal{E}(M) = \langle E(M), \pi, M, F, G\rangle$ is called a principal fiber bundle if and only if $F = G$ and if G has a differentiable, free, right action

$$R: E(M) \times G \to E(M) \tag{A1.11}$$

on the bundle space $E(M)$.

A right action satisfies $R(w, g_1 g_2) = R(R(w, g_1), g_2)$. Such an action is free, if and only if every element of G other than the identity, acts on $\mathcal{E}(M)$ without fixed points. More precisely, a right action $R: E(M) \times G \to E(M)$ is free, if and only if

$$\forall g \varepsilon G[(\exists w \varepsilon E(M)(R(w,g)=w)) \Longrightarrow (g=e)] \tag{A1.12}$$

A fiber fundle $\mathcal{E}(M)$ which is a principal bundle will be denoted by

$$\mathcal{P}(M) = \langle P(M), \pi_P, M, G\rangle \tag{A1.13}$$

where $P(M)$ is the total space, π_P is the projection map, M is the base space and G is the structure group of the bundle.

The principal fiber bundle plays a special role in the following sense: Given a principal bundle $\mathcal{P}(M)$ and some differentiable manifold F (that is, some representative space, not necessarily diffeomorphic to G, on which G acts differentiably and effectively on the left) then one can always construct the unique associated fiber bundle (AFB) $\mathcal{E}(M)$ with typical fiber F. Conversely, every fiber bundle $\mathcal{E}(M)$ is associated with an essentially unique principal bundle in this way.

To emphasise this unique relationship, it is useful to characterize the structure of any fiber bundle $\mathcal{E}(M)$ by writing

$$\mathcal{E}(M) = \langle E(M), \pi_E, M, F, \mathcal{P}(M) \rangle. \qquad \text{(A1.14)}$$

TANGENT BUNDLE AND PRINCIPAL BUNDLE OF LINEAR FRAMES

The above discussion may be usefully illustrated with the important examples of the principal bundle $\mathcal{L}(M)$ of linear frames and the associated bundle $\mathcal{T}(M)$ of tangent vectors of an n-dimensional manifold M.

Denote by $\mathcal{F}(M,\mathbb{R})$ the set of all differentiable functions $f: M \to \mathbb{R}$. A tangent vector $X_p \varepsilon T(M_p)$ is a function

$$X_p: \mathcal{F}(M,\mathbb{R}) \to \mathbb{R} \qquad \text{(A1.15)}$$

which for all $a_1, a_2 \varepsilon \mathbb{R}$ and for all $f_1, f_2 \varepsilon \mathcal{F}(M,\mathbb{R})$ satisfies

$$X_p(a_1 f_1 + a_2 f_2) = a_1 X_p(f_1) + a_2 X_p(f_2) \qquad \text{(A1.16)}$$

$$X_p(f_1 f_2) = f_1(p) X_p(f_2) + f_2(p) X_p(f_1)$$

where the sum and product of functions is defined pointwise. The formulas (A1.16) correspond to the rules for differentiating sums and products of functions.

For any coordinate chart (U,x), any function $f \epsilon F(M,\mathbb{R})$ may be decomposed in the following manner, namely,

$$\begin{tikzcd} M \supset U \arrow[rr, "f"] \arrow[dr, "x"'] & & \mathbb{R} \\ & x(U) \subset \mathbb{R}^n \arrow[ur, "F"'] & \end{tikzcd} \qquad \text{(A1.17)}$$

where $f = F \circ x$ on U. The maps

$$e_{ip} : F(M,\mathbb{R}) \rightarrow \mathbb{R}$$

defined by

$$e_{ip}(f) \equiv \frac{\partial F}{\partial x^i}(x(p)), \quad (i=1,2,\ldots,n) \qquad \text{(A1.18)}$$

satisfy (A1.16) and are therefore tangent vectors. The maps e_{ip} are frequently denoted by $\partial/\partial x^i_p$. They form a basis for $T(M_p)$. Thus, for any $X_p \epsilon T(M_p)$

$$X_p = X^i_p e_{ip} = X^i_p \frac{\partial}{\partial x^i_p} \qquad \text{(A1.19)}$$

where $X^i_p \epsilon \mathbb{R}$ are the components of the vector.

The total space TM of the tangent bundle $T(M) = \langle TM, \pi_T, M, \mathbb{R}^n, G\rangle$ is the set of all pairs (p, X_p), where $p \epsilon M$, and $X_p \epsilon T(M_p)$. The projection map π_T: $T(M) \rightarrow M$ is defined by $\pi_T(p, X_p) = p$ and the fiber over p is $T(M_p) = \pi^{-1}(p)$.

Let $\vec{e}_i$ denote the standard basis of the vector space $\mathbb{R}^n$. Define the map ψ_U: $T(U) \to \mathbb{R}^n$, where $T(U) = \pi_T^{-1}(U)$, by

$$\psi_U(p,X_p) = X_p^i \vec{e}_i . \tag{A1.20}$$

Then the restriction of ψ_U to $\psi_U|_p$: $T(M_p) \to \mathbb{R}^n$ and the map $\pi_T \times \psi$: $T(U) \to U \times \mathbb{R}^n$ are bijective. The topology and differentiable structure of $T(M)$ is defined by the atlas which consists of the family of charts $\{(T(U),(x \circ \pi_T) \times \psi)\}$ corresponding to an atlas $\{(U,x)\}$ for M. The maps π_T and ψ are then differentiable, and the maps $\pi_T \times \psi$ and $\psi_U|_p$ are diffeomorphisms.

Under a change of coordinate charts from (U,x) to $(\bar{U},\bar{x})$, the coordinate basis vectors are related by

$$\frac{\partial}{\partial x_p^i} = \frac{\partial \bar{\Lambda}^j}{\partial x^i}(x(p)) \frac{\partial}{\partial \bar{x}_p^i} \tag{A1.21}$$

where $\bar{\Lambda} \equiv \bar{x} \circ x^{-1}$: $x_\vdash(\bar{U} \cap U) \to \bar{x}_\vdash(\bar{U} \cap U)$. Consequently,

$$X_p = X_p^i \frac{\partial}{\partial x_p^i} = X_p^i \frac{\partial \bar{\Lambda}^j}{\partial x^i}(x(p)) \frac{\partial}{\partial \bar{x}_p^i} = \bar{X}_p^i \frac{\partial}{\partial \bar{x}_p^i} \tag{A1.22}$$

and

$$\bar{X}_p^i = \frac{\partial \bar{\Lambda}^i}{\partial x^j}(x(p)) X_p^i . \tag{A1.23}$$

Corresponding to this change of coordinate charts for M, the map ψ: $T(U) \to \mathbb{R}^n$ is replaced by $\bar{\psi}$: $T(\bar{U}) \to \mathbb{R}^n$, where

$$\overline{\psi}(p,X_p) = \overline{X}_p^i \vec{e}_i . \qquad \text{(A1.24)}$$

The change of representative from $\psi(p,X_p) = X_p^i \vec{e}_i$ to $\overline{\psi}(p,X_p) = \overline{X}_p^i \vec{e}_i$ is effected by a differentiable effective left action of GL(n) on the typical fiber $\mathbb{R}^n$ given by

$$\overline{X}_p^i = \frac{\partial \overline{\Lambda}^i}{\partial x^j}(x(p)) X_p^j .$$

Thus the structure group of $\mathcal{T}(M)$ is GL(n) and the induced transition map corresponding to a change of coordinate charts is the map

$$g: \overline{U} \cap U \to GL(n)$$

given by

$$g_{\overline{\psi}\psi}(p) = \frac{\partial \overline{\Lambda}^i}{\partial x^j}(x(p)) \vec{e}_i \otimes \vec{e}^{*j} \qquad \text{(A1.25)}$$

where $\vec{e}^{*i}$ is the basis for $\mathbb{R}^{*n}$ dual to the standard basis $\vec{e}_i$ of $\mathbb{R}^n$.

A <u>linear frame</u> is an n-tuple of linearly independent vectors or a basis $(X_{1p}, X_{2p}, \ldots, X_{np})$ of $T(M_p)$. The total space L(M) of the <u>principal bundle</u> of linear frames

$$\mathcal{L}(M) = \langle L(M), \pi_L, M, GL(n) \rangle \qquad \text{(A1.26)}$$

is the set of all pairs $(p,(X_{1p}, X_{2p}, \ldots, X_{np}))$. The projection map $\pi_L: L(M) \to M$ is defined by

$$\pi_L(p,(X_{1p}, X_{2p}, \ldots, X_{np})) = p \qquad \text{(A1.27)}$$

The set of linear frames $L(M_p)$ at $p\varepsilon M$, is the fiber at $p\varepsilon M$, and the total space for the portion over U is denoted by $L(U) = \pi_L^{-1}(U)$.

The map $\Psi: L(U) \to GL(n)$ is defined by

$$\Psi(p,(X_{1p},X_{2p},\ldots,X_{np})) = X^i_{jp}\vec{e}_i \otimes \vec{e}^{*j}$$

Then the bijective maps

$$\pi_L \times \Psi: L(U) \to U \times GL(n)$$

(one for each coordinate chart (U,x)), are used to define the topology and differentiable structure of $L(M)$. Under a change of coordinate charts, the representatives $\Psi(p,(X_{1p},X_{2p},\ldots,X_{np})) = X^i_{jp}\vec{e}_i \otimes \vec{e}^{*j}$ transform according to the analogue of (A1.23), namely,

$$\overline{X}^i_{jp} = \frac{\partial \overline{\Lambda}^i}{\partial x^k}(x(p))X^k_{ip}, \tag{A1.28}$$

which is a left action of $GL(n)$ on itself that is <u>free</u> —rather than just effective. The transition maps are again given by

$$g_{\overline{\psi}\psi}(p) = \frac{\partial \overline{\Lambda}^i}{\partial x^j}(x(p))\vec{e}_i \otimes \vec{e}^{*j},$$

and the differentiable, <u>free</u> action

$$R: L(M) \times GL(n) \to L(M)$$

is defined as

$$R((p,(X_{1p},X_{2p},\ldots,X_{np})),a) = (p,(X_{ip}a^i_1,\ldots,X_{ip}a^i_n) \tag{A1.29}$$

SYMMETRY

One of the important advantages of the fiber bundle formalism is that it supports the key distinctions between invariance and covariance, between symmetry and structure, between active and passive transformations.

As we already indicated, a geometric object is represented by an element of some representation space, the typical fiber space, which is ultimately some real coordinate space. The role of the structural group G of a bundle of geometric objects is to specify how much must be known about the relation between two coordinate systems (usually specified in terms of the number of derivatives of the required coordinate transformation function $\overline{\Lambda}=\overline{x}\circ x^{-1}$) in order to translate the description of a geometric object with respect to one coordinate system into the description with respect to another coordinate system. In fiber bundle language, this translation is effected by means of the action of the structural group on the appropriate typical fiber, the representation space of the geometric objects in question.

The geometric structure of a manifold is specified by means of one or more (local or global) cross section of the corresponding bundles of geometric objects. A symmetry of such a structure is an active transformation of the manifold which preserves its geometric structure. The set of such symmetries characterizes the degree of homogeneity of the geometric structure of the manifold. A precise discussion of the concept of symmetry requires the notion of a <u>bundle automorphism</u>.

DEFINITION A1.1: For any fiber bundle $E(M)$ (A1.14), a <u>bundle automorphism</u> is a pair of diffeomorphisms $\phi_M: M \to M$ and $\phi_{E(M)}: E(M) \to E(M)$ such that

$$\phi_M \circ \pi = \pi \circ \phi_{E(M)}. \tag{A1.30}$$

A bundle automorphism of $E(M)$ will simply be denoted by

$$\phi: E(M) \to E(M) \tag{A1.31}$$

Note that for every $p \varepsilon M$, the restriction of $\phi_{E(M)}$ maps the fiber $E(M_p)$ diffeomorphically onto the fiber $E(M_{\phi_M(p)})$.

DEFINTION A1.2: The transformation $\sigma^\phi: M \to E(M)$ of a cross section $\sigma: M \to E(M)$ under a bundle automorphism $\phi: E(M) \to E(M)$ is defined by

$$\sigma^\phi = \phi_{E(M)} \circ \sigma \circ \phi_M^{-1} \tag{A1.32}$$

The situation that obtains in the above definitions is summarized by the following commutative diagram.

$$\begin{tikzcd}
E(M) \arrow[r, shift left, "\phi_{E(M)}"] \arrow[d, "\pi"] & E(M) \arrow[l, shift left, "\phi^{-1}_{E(M)}"] \arrow[d, "\pi"'] \\
M \arrow[u, bend left=60, "\sigma"] \arrow[r, shift left, "\phi_M"] & M \arrow[l, shift left, "\phi_M^{-1}"] \arrow[u, bend right=60, "\sigma^\phi"']
\end{tikzcd} \tag{A1.33}$$

DEFINITION A1.3: A <u>global symmetry</u> for a geometric object field $\sigma: M \to E(M)$ is a bundle automorphism $\phi: E(M) \to E(M)$ such that $\sigma^{\phi} = \sigma$. The <u>global symmetry group</u> of a geometric object field is the set of such bundle automorphisms with the group product defined by the composition of bundle automorphisms, namely,

$$(\phi_M^{(1)}, \phi_M^{(1)}) \circ (\phi_{E(M)}^{(2)}, \phi_{E(M)}^{(2)}) = (\phi_M^{(1)} \circ \phi_M^{(2)}, \phi_{E(M)}^{(1)} \circ \phi_{E(M)}^{(2)}). \tag{A1.34}$$

The definition of a <u>local symmetry</u> is the same as that of a global symmetry except for the restriction of the domain of the maps $(\phi_M, \phi_{E(M)})$ to some local neighborhood U of M and E(U) of E(M). The targets of these maps are also restricted to $\phi_{M}(U)$ and $E(\phi_{M}(U))$. The <u>local symmetry pseudogroup</u> of a geometric object field is the set of such local symmetries with the product defined by (A1.34) subject to appropriate target and domain restrictions.

If only those local symmetries are considered which leave some $p \in U \subset M$ fixed, then the terms <u>local p-symmetry</u> and <u>local p-symmetry pseudogroup</u> will be used.

Finally, the restriction of a local p-symmetry to the fiber $E(M_p)$ is called a <u>microsymmetry</u> and the <u>microsymmetry group</u> is the set of such microsymmetry transformations.

2. JETS

Many geometric structures are first order structures. That is, the laws which govern the transformation of their representatives under a change of local coordinate charts depend only on the Jacobian of the coordinate transformation functions $\overline{\Lambda} = \overline{x} \circ x^{-1}$. The Riemannian, pseudo-Riemannian and conformal structures are first order structures.

An analysis of the Law of Inertia and the Principle of Equivalence however, involves investigating higher order geometric structures such as the affine and projective structures and the closely related acceleration and directing fields.

These structures are of second order. That is, the second order analogue of GL(n) is required for their transformations.

Analyses of higher order structures have been carried out using the standard formalism of T(M) and T(T(M)). For our purposes however, the simpler and much more natural description of higher order structures is in terms of the jet and jet bundle formalism of C. Ehresmann. We will employ that formalism in the construction of curve and path structures.

Conceptually, the elements of the second order jet bundle $J^2(\mathbb{R}_0, M)$ have a direct interpretation as second degree Taylor approximations to curves through a given point of M. The derivation of the coordinate parameter and active transformation laws merely involves the application of the chain law. In contrast, the elements of $T(T(M))$, which in jet language is denoted by $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$, are more complicated. The elements of the desired subbundle $J^2(\mathbb{R}_0, M)$ of $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$ must in the standard approach be selected by imposing the spray condition on the elements of $J^1(\mathbb{R}_0, J^1(\mathbb{R}_0, M))$.

Using jets, the analysis of the coordinate, parameter and active transformation laws of the relevant higher order geometric objects is therefore much more direct and intuitive. Moreover, the relationship between acceleration fields and the corresponding second order differential equations becomes transparent and the investigation of the associated path structures is likewise simplified.

Let $F(\mathbb{R}^n, \mathbb{R}^m)$ be the set of maps $\{f | f: \mathbb{R}^n \to \mathbb{R}^m\}$. Then the set $F(\mathbb{R}^n, \mathbb{R})$ is an algebra over $\mathbb{R}$ with scalar multiplication, addition and multiplication defined by

$$(\lambda f)(x) = \lambda f(x)$$

$$(f + g)(x) = f(x) + g(x) \qquad \text{(A2.1)}$$

$$(fg)(x) = f(x) \cdot g(x)$$

where $\lambda \varepsilon \mathbb{R}$, $f, g \varepsilon F(\mathbb{R}^n, \mathbb{R})$ and $x \varepsilon \mathbb{R}^n$.

The algebra of the set of functions $F(\mathbb{R}^n, \mathbb{R})$ is commutative and associative. The zero element is the zero function which maps every point $x \in \mathbb{R}^n$ into 0 and the unit element is the constant function which maps every point $x \in \mathbb{R}^n$ into 1.

The above construction applies equally well to the set of functions

$$F(M, \mathbb{R}) = \{f \mid f\colon M \to \mathbb{R}\}.$$

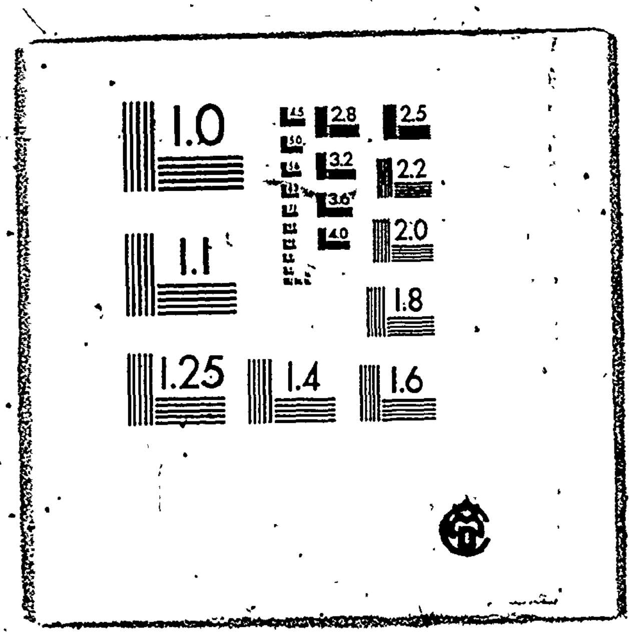
If M is a certain topological space, then the set of continuous functions $f\colon M \to \mathbb{R}$ is a subset of $F(M,\mathbb{R})$ and will be denoted by $C^0(M,\mathbb{R})$. If M is a C^r-manifold then the set of functions $f\colon M \to \mathbb{R}$ whose k-th order derivatives are continuous $(k \leq r)$ will be denoted by $C^k(M,\mathbb{R})$. The set of functions $f\colon M \to \mathbb{R}$ which are C^k in some open neighborhood U of $p \in M$ is denoted by $C^k(M_p,\mathbb{R})$.

Sets with distinguished base points, such as M_p, are said to be <u>pointed sets</u>. A map $f\colon M_p \to N_q$ between pointed sets such that $f(p) = q$ is said to be a <u>pointed map</u>.

If M and N are C^r-manifolds, then the set of $C^k (k \leq r)$ pointed maps $F(M_p,N_q)$ is denoted by $C^k(M_p,N_q)$ and $C^k(M_p,N_q) \subset C^k(M_p,N)$.

The set $C^k(M_p,\mathbb{R})$ is a subalgebra of the algebra structure on the set of functions $F(M,\mathbb{R})$. Moreover, the algebra on $C^k(M,\mathbb{R})$ is a subalgebra of the algebra

4 OF / DE 4



on $C^k(M_p, \mathbb{R})$ since the functions belonging to $C^k(M_p, \mathbb{R})$ need only be C^k in some open neighborhood of $p \varepsilon M$.

Let M be a C^∞-manifold of dimension n. Denote by Δ_p, $(p \varepsilon M)$, the subset $C^\infty(M, \mathbb{R})$ defined by

$$\Delta_p = \{f \varepsilon C^\infty(M, \mathbb{R}) \,|\, f(p) = 0,\ p \varepsilon M\}. \tag{A2.2}$$

Δ_p is an <u>ideal</u> of the algebra $C^\infty(M, \mathbb{R})$ and so are the powers Δ_p^k consisting of all linear combinations of k-products of elements of Δ_p. One then defines the <u>factor</u> algebra by

$$J^k(M_p, \mathbb{R}) = C^\infty(M, \mathbb{R})/\Delta_p^{k+1}. \tag{A2.3}$$

The canonical projection denoted by

$$j_p^k\colon C^\infty(M, \mathbb{R}) \to J^k(M_p, \mathbb{R}) \tag{A2.4}$$

is an <u>algebra homomorphism</u>. The image of $f \varepsilon C^\infty(M, \mathbb{R})$ under the projection homomorphism j_p^k is denoted by $j_p^k f$ and is called the <u>k-jet of f at $p \varepsilon M$</u>.

This definition of the k-jet of a continuous function f does not require the use of local coordinate charts. Of course its applicability is restricted to the sets of functions $C^\infty(M, \mathbb{R})$ that define an algebraic structure. In the more general cases an algebraic structure is not always available.

The equivalence relation is independent of the choice of coordinate charts as can be seen by applying the chain rule to the relation

$$\bar{y}\circ f\circ \bar{x}^{-1} = (\bar{y}\circ y^{-1})\circ(y\circ f\circ x^{-1})\circ(x\circ \bar{x}^{-1}). \qquad (A2.7)$$

The k-jet $j_p^k f$ of a map $f \in C^\infty(M,N)$ is the equivalence class to which f belongs.

The sets of k-jets of maps $f: M \to N$ with

i) source $p \in M$ and target $q \in N$

ii) source $p \in M$ and arbitrary target

iii) arbitrary source and target $q \in N$

iv) arbitrary source and arbitrary target are

respectively denoted by

$$\begin{aligned}
&a)\quad J^k(M_p,N_q) \\
&b)\quad J^k(M_p,N) = \bigcup_{q\in N} J^k(M_p,N_q) \\
&c)\quad J^k(M,N_q) = \bigcup_{p\in M} J^k(M_p,N_q) \\
&d)\quad J^k(M,N) = \bigcup_{p\in M}\ \bigcup_{q\in N} J^k(M_p,N_q)
\end{aligned} \qquad (A2.8)$$

The canonical projection is denoted by the map

$$j_p^k\colon C^\infty(M,N) \to J^k(M_p,N). \qquad (A2.9)$$

Since $j_p^k f \subset j_p f$, one may also use j_p^k to denote the projection

The equivalence relation is independent of the choice of coordinate charts as can be seen by applying the chain rule to the relation

$$\bar{y}\circ f\circ \bar{x}^{-1} = (\bar{y}\circ y^{-1})\circ(y\circ f\circ x^{-1})\circ(x\circ \bar{x}^{-1}). \tag{A2.7}$$

The k-jet $j_p^k f$ of a map $f \varepsilon C^\infty(M,N)$ is the equivalence class to which f belongs.

The sets of k-jets of maps $f: M \to N$ with

i) source $p \varepsilon M$ and target $q \varepsilon N$

ii) source $p \varepsilon M$ and arbitrary target

iii) arbitrary source and target $q \varepsilon N$

iv) arbitrary source and arbitrary target are

respectively denoted by

$$\begin{aligned}
&\text{a)} \quad J^k(M_p, N_q) \\
&\text{b)} \quad J^k(M_p, N) = \bigcup_{q\varepsilon N} J^k(M_p, N_q) \\
&\text{c)} \quad J^k(M, N_q) = \bigcup_{p\varepsilon M} J^k(M_p, N_q) \\
&\text{d)} \quad J^k(M,N) = \bigcup_{p\varepsilon M} \bigcup_{q\varepsilon N} J^k(M_p, N_q)
\end{aligned} \tag{A2.8}$$

The canonical projection is denoted by the map

$$j_p^k: C^\infty(M,N) \to J^k(M_p, N). \tag{A2.9}$$

Since $j_p^k f \subset j_p f$, one may also use j_p^k to denote the projection

$$j_p^k\colon J(M_p,N) \to J^k(M_p,N), \tag{A2.10}$$

and write

$$j_p^k(j_p f) = j_p^k f. \tag{A2.11}$$

There are well defined projection maps from $J^k(M,N)$ onto M, N and M×N which are defined by

$$\sigma(j_p^k f) = p$$

$$\tau(j_p^k f) = f(p) \tag{A2.12}$$

$$\sigma\times\tau(j_p^k f) = (p,f(p)).$$

If M, L and N are C^∞-manifolds, then the composition map

$$\circ\colon C^\infty(M,L)\times C^\infty(L,N)$$

defined by

$$g\circ f(p) = g(f(p))$$

induces a number of compositions between germs and jets, namely,

$$j_p(g\circ f) = j_{f(p)}g\circ j_p f$$

$$j_p^k(g\circ f) = j_{f(p)}^k g\circ j_p f \tag{A2.13}$$

$$= j_{f(p)}g\circ j_p^k f$$

$$= j_{f(p)}^k g\circ j_p^k f$$

3. BUNDLES OF HIGHER ORDER CURVE ELEMENTS AND HIGHER ORDER FRAMES

Consider the set of maps $F(\mathbb{R},M)$ called curves. By curve elements we mean, rougly speaking, Taylor polynomial approximations at $p\varepsilon M$ to curves $\gamma\varepsilon F(\mathbb{R},M)$ through $p\varepsilon M$.

We define equivalence classes for curves by applying Taylor polynomial approximations to the coordinate representatives of maps $F(\mathbb{R},M)$.

DEFINITION A3.1: Any two curves $\gamma,\hat{\gamma}\varepsilon F(\mathbb{R},M)$ through a given point $p\varepsilon M$ are k-jet equivalent at $p\varepsilon M$, if and only if

i) $\gamma(0) = \hat{\gamma}(0) = p$.

ii) for any chart $(U,x)_p$, the curves $x\circ\gamma$, $x\circ\hat{\gamma}$ through $x(p)\varepsilon\mathbb{R}^n$ have the same derivatives at $x(p)\varepsilon\mathbb{R}^n$ up to and including order k.

Hence by k-th order curve elements we mean equivalence classes of k-jet equivalent curves. We shall also say that k-jet equivalent curves have contact of order k at $p\varepsilon M$.

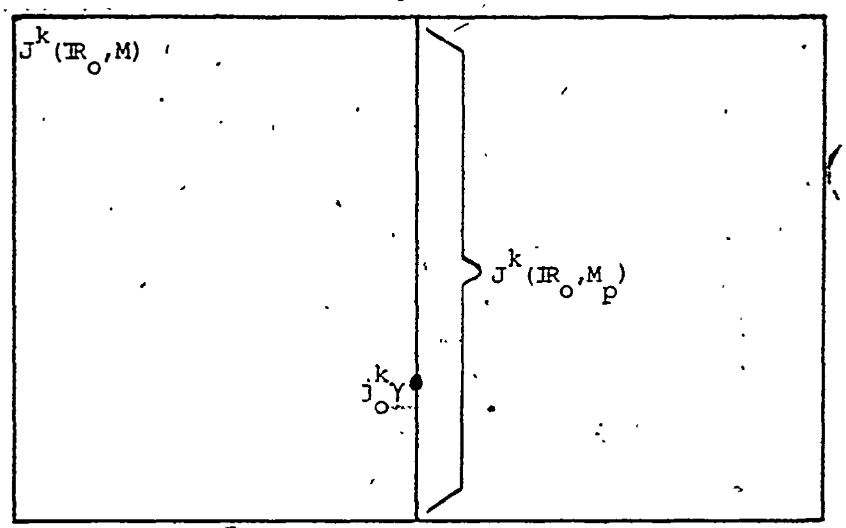
The equivalence class of k-jet equivalent curves at $p\varepsilon M$ will be denoted by $j_0^k\gamma$. The equivalence class of 1-jet equivalent curves is denoted by $j_0^1\gamma$ and corresponds to all curves through p whose first order contact or tangency at p is characterized by a unique tangent vector $X_p\varepsilon TM_p$. Hence, according to this approach, a tangent vector at a point $p\varepsilon M$ is definable in terms of a set of curves through $p\varepsilon M$ and a coordinate map, such that the images of these curves have the same tangent at $x(p)\varepsilon \mathbb{R}^n$. The set of all such equivalence classes at $p\varepsilon M$ is denoted by $J^1(\mathbb{R}_0, M_p)$ and the set of all pairs $(p, j_0^1\gamma)$ is the space

$$J^1(\mathbb{R}_0, M) = \bigcup_{p\varepsilon M} J^1(\mathbb{R}_0, M_p). \tag{A3.1}$$

$J^1(\mathbb{R}_0, M_p)$ and $J^1(\mathbb{R}_0, M)$ are isomorphic to TM_p and TM respectively.

It follows from (A2.7) that the equivalence relation of the maps $F(\mathbb{R}_0, M_p)$ does not depend on the particular chart $(U,x)_p$. The Taylor polynomial corresponding to $j_0^k\gamma$ depends of course on the chart $(U,x)_p$. For any given chart however, the derivatives $\gamma_1^i, \gamma_2^i, \ldots, \gamma_k^i$ of the functions $x^i \circ \gamma \equiv \gamma^i$ at the given point uniquely determine, and are uniquely determined by $j_0^k\gamma$. They may thus be used as coordinates for $j_0^k\gamma$.

Some of the preceding notions are illustrated in the diagram below.


$J^k(\mathbb{R}_o, M)$
$J^k(\mathbb{R}_o, M_p)$
$j_o^k\gamma$

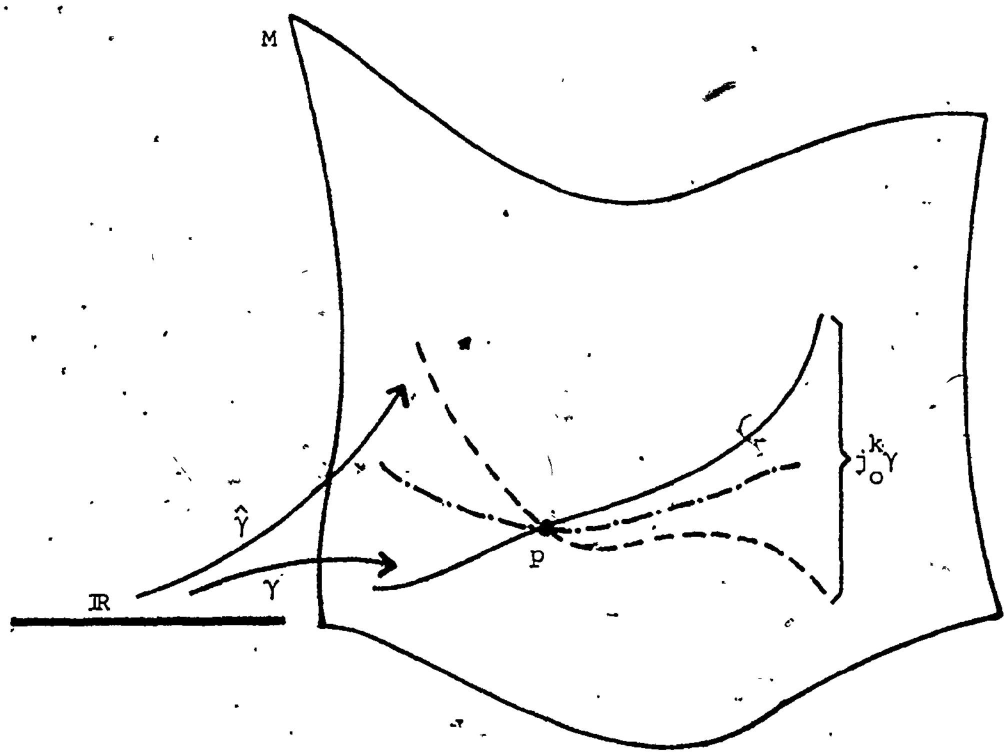

M
$\hat{\gamma}$
γ
$\mathbb{R}$
p
$j_o^k\gamma$

FIGURE A4

HIGHER ORDER ANALOGUES OF GL(n)

It is convenient at this point to introduce some special notation. Set

$$L^k_s(\mathbb{R}^n_{\vec{0}}) \equiv J^k(\mathbb{R}^s_{\vec{0}}, \mathbb{R}^n_{\vec{0}}). \tag{A3.2}$$

Denote by $G^k_{s,n}$ the subset of $L^k_s(\mathbb{R}^n_{\vec{0}})$ consisting of jets of maximal rank. For $s=n$, set $G^k_{n,n} \equiv G^k_n$. Clearly, the set of all diffeomorphisms $a: \mathbb{R}^n \to \mathbb{R}^n$ which preserves the origin (that is, $a(\vec{0}) = \vec{0}$) form a group under map composition $c = a \circ b$. It is a straightforward matter to evaluate the Taylor polynomial of degree k of any such map. For example, for $k = 3$

$$\tilde{t}^i = a^i_j t^j + \frac{1}{2!} a^i_{jk} t^j t^k + \frac{1}{3!} a^i_{jk\ell} t^j t^k t^\ell \tag{A3.3}$$

where the coefficients a^i_j, a^i_{jk}, $a^i_{jk\ell}$ are the partial derivatives at $\vec{0}$ of the n maps $a^i: \mathbb{R}^n \to \mathbb{R}$ which define a. Since a is a diffeomorphism $\det(a^i_j) \neq 0$. Denote by $j^k_{\vec{0}}a$ the equivalence class of all those diffeomorphisms a which have the same Taylor polynomial of degree k at $\vec{0}$. The set of such equivalence classes is denoted by G^k_n and forms a Lie group with the group product defined by

$$j^k_{\vec{0}}a \circ j^k_{\vec{0}}b = j^k_{\vec{0}}(a \circ b). \tag{A3.4}$$

The group of elements may be coordinatized by the coefficients of the Taylor polynomials which characterize the equivalence classes. For example, for $k = 3$,

$$j_0^3 a = (a_j^i, a_{jk}^i, a_{jk\ell}^i). \tag{A3.5}$$

If $j_0^3 c = j_0^3 a \circ j_0^3 b$, then the group coordinates of $j_0^3 c$ are given in terms of those for $j_0^3 a$ and $j_0^3 b$ by

$$\begin{aligned} c_j^i &= a_r^i b_j^r \\ c_{jk}^i &= a_r^i b_{jk}^r + a_{rs}^i b_j^r b_k^s \qquad\qquad (A3.6) \\ c_{jk\ell}^i &= a_r^i b_{jk\ell}^r + a_{rs}^i (b_j^r b_{k\ell}^s + b_k^r b_{\ell j}^s + b_\ell^r b_{jk}^s) \\ &\quad + a_{rst}^i b_j^r b_k^s b_\ell^t. \end{aligned}$$

These formulas may be obtained either by applying the chain rule to evaluate the partial derivatives of the composite function $a \circ b$, or by taking the Taylor polynomial which represents $j_0^k b$ and substituting it into the Taylor polynomial which represents $j_0^k a$ and discarding those terms of degree greater than k. The result is a Taylor polynomial which represents $j_0^k c$.

One can readily verify that the identity element of G_n^3 is

$$j_0^3 id_{\mathbb{R}^n} = (\delta^i_j, 0, 0) \tag{A3.7}$$

and that the element $(j_0^3 a)^{-1}$, inverse to $j_0^3 a$, is $j_0^3 a^{-1}$ where

$$(j_0^3 a^{-1})^i_j = a^{-1i}_{\ j}$$

$$(j_0^3 a^{-1})^i_{jk} = -a^{-1i}_{\ r} a^r_{st} a^{-1s}_{\ j} a^{-1t}_{\ k} \tag{A3.8}$$

$$(j_0^3 a^{-1})^i_{jk\ell} = -a^{-1i}_{\ r} a^r_{stu} a^{-1s}_{\ j} a^{-1t}_{\ k} a^{-1u}_{\ \ell}$$

$$+ a^{-1i}_{\ r} a^{-1m}_{\ s} (a^r_{mt} a^s_{uv} + a^r_{mu} a^s_{vt} + a^r_{mv} a^s_{tu}) a^{-1t}_{\ j} a^{-1u}_{\ k} a^{-1v}_{\ \ell}$$

where the matrix $(a^{-1i}_{\ j})$ is the inverse of the matrix (a^i_j). Clearly, G^1_n is just the familiar general linear group $GL(n)$.

PRINCIPAL BUNDLE OF K-ORDER FRAMES

Analogous to the notation introduced by (A3.2) we set

$$L^k_s(M_p) \equiv J^k(\mathbb{R}^s_0, M_p). \tag{A3.9}$$

The manifold $L^k_s(M_p)$ denotes the s^k-speeds at $p \in M$ and we denote by $H^k_s(M_p)$, where

$$H^k_s(M_p) \subset L^k_s(M_p). \tag{A3.10}$$

the submanifold of s^k-frames. $H^k_s(M_p)$ consists of jets of maximal rank. Normally, $s < n$, but if $s = n$ we shall simply write $H^k(M_p)$ for the set of n^k-frames. The maximal rank condition implies that

$$H^k_s(M_p) = L^k_s(M_p) \setminus \{(p, j^k_{\vec{0}}\gamma) \mid j^1_{\vec{0}}\gamma = 0\}. \qquad \text{(A3.11)}$$

Let B be an open neighborhood of $\vec{0} \varepsilon \mathbb{R}^n$ and let $h\colon B \to h(B) \subset M$ be a local diffeomorphism such that $h(\vec{0}) = p$. Such a map provides an image of the coordinate grid of $\mathbb{R}^n$ in M. Given any chart $(U,x)_p$ and setting $x^i \circ h \equiv h^i$, then the coefficients of the Taylor polynomial $j^k_{\vec{0}}(x \circ h)$ are given by

$$j^k_{\vec{0}} h^i = (h^i_{j_1}, h^i_{j_1 j_2}, \ldots, h^i_{j_1 j_2 \ldots j_k}) \qquad \text{(A3.11)}$$

and serve as local coordinates for $j^k_{\vec{0}} h \varepsilon H^k(M_p)$. These coordinates transform under a change of coordinate charts according to formulas which for $k = 3$ are similar to the formulas (A3.6) with c, a and b replaced by $\bar{h}$, $\bar{\Lambda}$ and h respectively.

The projection $\pi_{H^k}\colon H^k(M) \to M$ is defined as usual. The group G^k_n acts freely on $H^k(M)$ on the right according to

$$\begin{aligned} R(j^k_{\vec{0}}h, j^k_{\vec{0}}a) &= j^k_{\vec{0}}a(j^k_{\vec{0}}h) \\ &= j^k_{\vec{0}}(h \circ a) \end{aligned} \qquad \text{(A3.11)}$$

$$= j_0^k \tilde{h}.$$

Again, for $k=3$, the formulas which represent the right free action (A3.11) are similar to (A3.6) with $\tilde{h}$, h and a replacing c, a and b.

The <u>principal bundle of n^k-frames</u> is the structure

$$H^k(M) = \langle H^k(M),\ \pi_H k,\ M,\ G_n^k \rangle. \tag{A3.12}$$

The principal bundle $H^1(M)$ is isomorphic to the bundle of linear frames which is customarily denoted by $L(M)$.

ASSOCIATED FIBER BUNDLE OF K-ORDER CURVE ELEMENTS

For a given coordinate chart $(U,x)_p$ for M, the locally trivializing maps may be used to give the sets $L_s^k(M)$ and $H_s^k(M)$ their usual topological and differential structures. For example, for the set of k-order curve elements $L_1^k(M) = J^k(\mathbb{R}_0, M)$, the map ψ_U: $L_1^k(U) \to L_1^k(\mathbb{R}_0^n)$ is defined by

$$\psi_U(j_0^k \gamma) = j_0^k(x \circ \gamma) \tag{A3.13}$$

and the set $L_1^k(M)$ is given a topology and a differentiable structure by requiring that the map

$$x \circ \pi_{L_1^k} \times \psi_U\colon\ L_1^k(U) \to x(U) \times L_1^k(\mathbb{R}_0^n) \tag{A3.14}$$

are diffeomorphisms.

The associated fiber bundle of k-order curve elements is the structure

$$\tilde{L}_1^k(M) = \langle L_1^k(M),\ \pi_{L_1^k},\ M,\ L_1^k(\mathbb{R}_0^n),\ H^k(M)\rangle, \qquad (A3.15)$$

where $H^k(M)$ denotes the principal bundle of n^k-frames.

4. ACCELERATION FIELDS

Consider the curve

$$\gamma^k : I \to J^k(\mathbb{R}_0, M) = L_1^k(M) \tag{A4.1}$$

where $I \subset \mathbb{R}$ and $k \varepsilon \{0,1,2,\ldots\}$. In terms of local coordinates we have

$$\gamma^k(s) = (\gamma_0^{ki}(s);\ \gamma_1^{ki}(s),\ \gamma_2^{ki}(s),\ldots,\ \gamma_k^{ki}(s)) \tag{A4.2}$$

where $s \varepsilon \mathbb{R}$ and $\gamma_a^{ki}(s) \equiv d^a x^i \circ \gamma(s)/ds^a$ for some $\gamma: I \to M$ which in general depends on $p = \gamma(s) \varepsilon M$.

DEFINITION A4.1: γ^k is a <u>special curve</u> if and only if for all $s \varepsilon \mathbb{R}$

$$\gamma_a^{ki}(s) = \dot{\gamma}_{a-1}^{ki}(s); \quad 0 \leq a \leq k. \tag{A4.3}$$

DEFINITION A4.2: A special curve γ^k is called the <u>k-order lift</u> of the curve $\pi_{L_1^k} \circ \gamma^k : \mathbb{R} \to M$.

If $\gamma = \pi_{L_1^k} \circ \gamma^k$ then one writes

$$j^k(\gamma) = \gamma^k \tag{A4.4}$$

to denote the k-order lift of γ.

It should be noted that the relations $\gamma_a^{ki}(s) = \dot{\gamma}_{a-1}^{ki}(s)$, for all $s \varepsilon \mathbb{R}$, do not hold in general, since the coordinates γ_a^{ki}, γ_1^{ki}, etc., are defined as derivatives only at a point.

Unless required for clarity, the coordinates of the base point and the superscript k denoting the order of the jet, will be suppressed; and we shall simply write

$$j_0^k\gamma = (\gamma_1^i, \gamma_2^i, \ldots, \gamma_k^i). \tag{A4.5}$$

The transformation law for the local coordinates γ_a^i is obtained from the formula

$$\begin{aligned} j_0^k(\bar{x}\circ\gamma) &= j_0^k(\bar{x}\circ x^{-1}\circ x\circ\gamma) \\ &= j_{x(p)}^k\bar{\Lambda}\circ j_0^k(x\circ\gamma). \end{aligned} \tag{A4.6}$$

Let $(U,x)_p$ and $(\bar{U},\bar{x})_p$ be the charts at $p\varepsilon M$. By application of the chain rule the corresponding coordinates

$$(\gamma_1^i),\ (\bar{\gamma}_1^i)$$

$$(\gamma_1^i, \gamma_2^i),\ (\bar{\gamma}_1^i, \bar{\gamma}_2^i)$$

$$(\gamma_1^i, \gamma_2^i, \gamma_3^i),\ (\bar{\gamma}_1^i, \bar{\gamma}_2^i, \bar{\gamma}_3^i)$$

of the 1, 2 and 3-jets are related by

$$\bar{\gamma}_1^i = \bar{\Lambda}_j^i \gamma_1^j$$

$$\bar{\gamma}_2^i = \bar{\Lambda}_j^i \gamma_2^j + \bar{\Lambda}_{jk}^i \gamma_1^j \gamma_1^k \qquad \text{(A4.7)}$$

$$\bar{\gamma}_3^i = \bar{\Lambda}_j^i \gamma_3^j + 3\bar{\Lambda}_{jk}^i \gamma_2^i \gamma_1^k + \bar{\Lambda}_{jk\ell}^i \gamma_1^j \gamma_1^k \gamma_1^\ell .$$

The $\bar{\Lambda}_j^i$, $\bar{\Lambda}_{jk}^i$, $\bar{\Lambda}_{jk\ell}^i$ denote the partial derivatives of $\bar{\Lambda} = \bar{x} \circ x^{-1}$ at $x(p) \varepsilon \mathbb{R}^n$, where $\Lambda = x \circ \bar{x}^{-1}$ and $\Lambda \circ \bar{\Lambda} = \bar{\Lambda} \circ \Lambda = id$. The $\bar{\Lambda}_j^i \varepsilon GL(n) \equiv G_n^1$, $(\bar{\Lambda}_j^i, \bar{\Lambda}_{jk}^i) \varepsilon G_n^2$ and $(\bar{\Lambda}_j^i, \bar{\Lambda}_{jk}^i, \bar{\Lambda}_{jk\ell}^i) \varepsilon G_n^3$ are the 1, 2 and 3-jets of $\bar{\Lambda}$ at $x(p) \varepsilon \mathbb{R}^n$ respectively.

Consider again the associated fiber bundle of k-order curve elements, namely,

$$L_1^k(M) = \langle L_1^k(M),\ \pi_{L_1^k},\ M,\ L_1^k(\mathbb{R}_0^n),\ H^k(M) \rangle .$$

DEFINITION A4.3: An <u>acceleration field</u> is a map $A: L_1^1(M) \to L_1^2(M)$ such that $\pi_{L_1^2}^{L_1^1} \circ A$ is the identity map on $L_1^1(M)$, where $\pi_{L_1^2}^{L_1^1}: L_1^2(M) \to L_1^1(M)$ is the natural projection obtained by dropping the second order term of the Taylor polynomial.

We are primarily interested in the mathematical characterization of <u>physical</u> acceleration fields. It is clear that Definition A4.3 is too wide for this purpose since it does not exclude

$$j_s^1 \gamma = \pi_{L_1^k}^{L_1^1}(j_s^k \gamma) = 0 \qquad \text{(A4.8)}$$

If (A4.8) obtains, then the motion represented by γ temporarily stops at s. Since time always "flows", curves in spacetime which are world lines of material particles must have $j^1_s\gamma \neq 0$ for all $s \in \mathbb{R}$. The open fiber submanifold of $L^k_1(M)$ consisting of k-jets of maps $\gamma: \mathbb{R} \to M$ which everywhere have maximal rank (i.e., 1) is denoted as in (A3.11) by $H^k_1(M)$.

DEFINITION A4.4: A physical acceleration field is a map $A: H^1_1(M) \to H^2_1(M)$ such that $\pi^{H^1_1}_{H^2_1} \circ A$ is the identity map on $H^1_1(M)$ where $\pi^{H^1_1}_{H^2_1}: H^2_1(M) \to H^1_1(M)$ is the natural projection obtained by dropping the second order term of the Taylor polynomial.

In terms of local coordinates (x^i, γ^i_1) for $H^1_1(M)$ and $(x^i, \gamma^i_1, \gamma^i_2)$ for $H^2_1(M)$, such a field is given by

$$A(x^i(p), \gamma^i_1) = (x^i(p), \gamma^i_1, A^i_2(x^i(p), \gamma^i_1)) \qquad \text{(A4.9)}$$

where $A^i_2(x^i(p), \gamma^i_1)$ are n functions of the 2n coordinates $(x^i(p), \gamma^i_1)$.

For any curve $\gamma \in F(\mathbb{R}, M)$, the curves $j^1\gamma: \mathbb{R} \to H^1_1(M)$ and $j^2\gamma: \mathbb{R} \to H^2_1(M)$, such that

$$\left.\begin{aligned} j^1\gamma(s) &= j^1_s\gamma \\ j^2\gamma(s) &= j^2_s\gamma, \end{aligned}\right\} \quad \forall s \in \mathbb{R}$$

are first and second order lifts of γ (Definition A4.2

and (A4.4)).

A curve $\gamma: \mathbb{R} \to M$ is a solution of the differential equation defined by A, if and only if

$$A \circ j^1\gamma = j^2\gamma, \tag{A4.10}$$

which, in terms of local coordinates may be written as

$$\frac{d^2\gamma^i}{ds^2}(s) = A_2^i(\gamma^i(s), \frac{d\gamma^i}{ds}(s)) \tag{A4.11}$$

By the fundamental existence theorem, if an element $(\gamma^i(t_0), \dot{\gamma}^i(t_0)) \varepsilon H_1^1(M)$ is specified for some initial value $t_0 \varepsilon \mathbb{R}$, then $A \circ j^1\gamma = j^2\gamma$ determines a unique curve $\gamma: \mathbb{R} \to M$.

$A_\vdash(H_1^1(M))$ is a 2n-dimensional submanifold of the 3n-dimensional manifold $H_1^2(M)$, and the lifts $j^2\gamma$ of the solution curves γ are curves in the 2n-dimensional submanifold.

DEFINITION A4.4: An acceleration field is called geodesic, and is denoted by

$$\Gamma: H_1^1(M) \to H_1^2(M),$$

if and only if, for all $p \varepsilon M$, there exists some local system of coordinates $(\bar{x}^i(p), \bar{\gamma}_1^i, \bar{\gamma}_2^i)$, with respect to which at $p \varepsilon M$,

$$\overline{\Gamma}(\overline{x}^i(p),\ \overline{\gamma}_1^i) = (\overline{x}^i(p),\ \overline{\gamma}_1^i,\ 0)$$

or (A4.12)

$$\overline{\Gamma}_2^i(\overline{x}^i(p),\ \overline{\gamma}_1^i) = 0.$$

This definition is a modern formulation of Weyl's definition of a symmetric linear connection.

THEOREM A4.1: An acceleration field Γ is geodesic if and only if relative to <u>any</u> local system of coordinates $(x^i(p),\ \gamma_1^i,\ \gamma_2^i)$

$$\Gamma_2^i(x^i(p),\ \gamma_1^i) = -\Gamma_{jk}^i(x^i(p))\gamma_1^j\gamma_1^k. \qquad \text{(A4.13)}$$

The proof of the theorem is readily obtained by using the transformation law (A4.7).

5. DIRECTION BUNDLES

A curve in M is a map $\gamma\varepsilon F(\mathbb{R},M)$ whose domain has the standard structure of an algebraic field. The only automorphism μ: $\mathbb{R}\to\mathbb{R}$ which preserves this field structure is the identity. The points on the one-dimensional submanifold $\gamma_{\vdash}(\mathbb{R})$ of M are therefore uniquely labeled.

Since the choice of the parametric origin is often not significant, translations are usually allowed and the structure of $\mathbb{R}$ is weakened to a one-dimensional Euclidean space. The points of a path, however, are not labeled in any particular manner. A path is the image set $\gamma_{\vdash}(\mathbb{R})$ of a curve viewed simply as a one-dimensional submanifold of M. A path in M is an equivalence class of curves γ: $\mathbb{R}\to M$, any two of which are related by $\hat{\gamma}=\gamma\circ\mu$, where μ: $\mathbb{R}\to\mathbb{R}$ is a diffeomorphism.

Since a path ξ is an equivalence class of curves, a path element of order k will be called a k-direction. As before, we restrict our attention to the open fiber submanifold $H_1^k(M)$ of k-jets of maps γ: $\mathbb{R}\to M$ which have maximal rank everywhere (see (A3.11)).

DEFINITION A5.1: Any two curves γ, $\hat{\gamma}$ are k-direction equivalent at $p\varepsilon M$, if and only if there exists a diffeomorphism μ: $\mathbb{R}\to\mathbb{R}$ such that

$$j_s^k\hat{\gamma} = j_s^k(\gamma\circ\mu) = j_{\mu(s)}^k\gamma\circ j_s^k\mu \tag{A5.1}$$

where $\hat{\gamma}(s) = \gamma(\mu(s)) = p$. An equivalence class of k-direction equivalent curves at $p\varepsilon M$ is called a k-direction at $p\varepsilon M$.

A k-direction may also be regarded as an equivalence class of paths. Let ξ be a path through $p\varepsilon M$. Choose any $\gamma\varepsilon\xi = [\gamma]$ such that $\gamma(0) = p$ and $j_0^1\gamma \neq 0$. For every $\hat{\gamma}\varepsilon j_0^k\gamma$, there corresponds a path, namely, $\hat{\xi} = \hat{\gamma}_{\vdash}(\mathbb{R})$. Then the set of all such paths is a k-direction at $p\varepsilon M$ and will be denoted by $j_p^k\xi$.

For a given choice of local coordinates (U,x) in M, there corresponds to each $j_p^k\xi$ an equivalence class of Taylor polynomials of degree k. For the case $k = 3$, the coefficients of any two such Taylor polynomials are according to (A5.1) related by

$$\hat{\gamma}_1^i = (D\mu)\gamma_1^i$$

$$\hat{\gamma}_2^i = (D\mu)^2\gamma_2^i + (D^2\mu)\gamma_1^i \tag{A5.2}$$

$$\hat{\gamma}_3^i = (D\mu)^3\gamma_3^i + 3(D^2\mu)(D\mu)\gamma_2^i + (D^3\mu)\gamma_1^i$$

where $D\mu$, $D^2\mu$ and $D^3\mu$ are the derivatives of μ at $0\varepsilon\mathbb{R}$.

The relation $j_0^k\hat{\gamma} = j_0^k\gamma \circ j_0^k\mu$ defines a right action of the group G_1^k on the space $H_1^k(M_p)$ of curve elements of order k at $p\varepsilon M$. The equivalence classes at $p\varepsilon M$ defined by this action are called k-directions at $p\varepsilon M$ and the space of these k-directions will be denoted by .

$$\mathbb{D}_1^k(M_p) \equiv H_1^k(M_p)/G_1^k. \tag{A5.3}$$

Since for all $\gamma\varepsilon\xi$, $j_0^1\gamma \neq \theta$, at least one of the components γ_1^i is not zero. Moreover, μ: $\mathbb{R}_0 \to \mathbb{R}_0$ is a diffeomorphism and hence $D\mu \neq 0$. Consequently the manifold $\mathbb{D}_1^k(M_p) = H_1^k(M_p)/G_1^k$ may be covered by n-coordinate charts defined as follows: for $b\varepsilon\{1,2,\ldots,n\}$ choose $D\mu$ so that $\hat{\gamma}_1^b = 1$ and choose $D^r\mu$ for each $r\varepsilon\{2,3,\ldots,k\}$ such that $\hat{\gamma}_r^b = 0$. Define $\xi_r^\alpha = \hat{\gamma}_r^\alpha$ for $r\varepsilon\{1,2,\ldots,k\}$ and $\alpha \neq b$.

For the special case $k=3$, the coordinates of $j_p^3\xi$ in the chart with $b=n$ are determined by the coordinates of $j_0^3\gamma$ by

$$\begin{aligned}
\xi_1^\alpha &= \frac{\gamma_1^\alpha}{\gamma_1^n} = \frac{dx^\alpha}{dx^n} \\
\xi_2^\alpha &= \frac{\gamma_2^\alpha\gamma_1^n - \gamma_2^n\gamma_1^\alpha}{(\gamma_1^n)^3} = \frac{d^2x^\alpha}{(dx^n)^2} \\
\xi_3^\alpha &= \frac{\gamma_3^\alpha}{(\gamma_1^n)^3} - 3\frac{\gamma_2^n}{(\gamma_1^n)^2}\,\frac{\gamma_2^\alpha\gamma_1^n - \gamma_2^n\gamma_1^\alpha}{(\gamma_1^n)^3} - \frac{\gamma_3^n}{(\gamma_1^n)^3}\,\frac{\gamma_1^\alpha}{\gamma_1^n} = \frac{d^3x^\alpha}{(dx^n)^3}.
\end{aligned} \tag{A5.4}$$

Note that in spacetime, the coordinates of the k-direction $j_p^k\xi$ are just the first k-derivatives of the space coordinates with respect to the time coordinate along some world line path. Thus ξ_1^α and ξ_2^α are customarily referred to as <u>three velocity</u> and <u>three acceleration</u> respectively. That $j_p^k\xi$ has the coordinates ξ_r^α will be indicated by

$$j_p^k\xi = (\xi_1^\alpha, \xi_2^\alpha, \ldots, \xi_k^\alpha). \tag{A5.5}$$

The law by which the coordinates of a k-direction transform under a change of coordinate chart from (U,x) to $(\overline{U},\overline{x})$ is obtained by using the transformation law for the γ_r^i together with the formulas which define the ξ_r^α in terms of the γ_r^i. For $k = 3$, the expressions (A4.7) and (A5.4) give

$$\overline{\xi}_1^\alpha = \frac{\overline{\Lambda}_n^\alpha + \overline{\Lambda}_\beta^\alpha \xi^\beta}{\overline{\Lambda}_n^n + \overline{\Lambda}_\gamma^n \xi_1^\gamma}$$

$$\begin{aligned}\overline{\xi}_2^\alpha = {} & \frac{\overline{\Lambda}_\rho^\alpha \xi_2^\rho + \overline{\Lambda}_{\rho\sigma}^\alpha \xi_1^\rho \xi_1^\sigma + 2\overline{\Lambda}_{n\rho}^\alpha \xi_1^\rho + \overline{\Lambda}_{nn}^\alpha}{(\overline{\Lambda}_n^n + \overline{\Lambda}_\beta^n \xi_1^\beta)^2} \\ & - \overline{\xi}_1^\alpha \, \frac{\overline{\Lambda}_\rho^n \xi_2^\rho + \overline{\Lambda}_{\rho\sigma}^n \xi_1^\rho \xi_1^\sigma + 2\overline{\Lambda}_{n\rho}^n \xi_1^\rho + \overline{\Lambda}_{nn}^n}{(\overline{\Lambda}_n^n + \overline{\Lambda}_\beta^n \xi_1^\beta)^2}\end{aligned} \tag{A5.6}$$

$$\begin{aligned}\overline{\xi}_3^\alpha = {} & \frac{\overline{\Lambda}_\rho^\alpha \xi_3^\rho + 3\overline{\Lambda}_{\rho\sigma}^\alpha \xi_2^\rho \xi_1^\sigma + 3\overline{\Lambda}_{n\rho}^\alpha \xi_2^\rho + \overline{\Lambda}_{\rho\sigma\tau}^\alpha \xi_1^\rho \xi_1^\sigma \xi_1^\tau + 3\overline{\Lambda}_{n\rho\sigma}^\alpha \xi_1^\rho \xi_1^\sigma + 6\overline{\Lambda}_{nn\rho}^\alpha \xi_1^\rho + \overline{\Lambda}_{nnn}^\alpha}{(\overline{\Lambda}_n^n + \overline{\Lambda}_\beta^n \xi_1^\beta)^3} \\ & - 3\overline{\xi}_2^\alpha \, \frac{\overline{\Lambda}_\rho^n \xi_2^\rho + \overline{\Lambda}_{\rho\sigma}^n \xi_1^\rho \xi_1^\sigma + 2\overline{\Lambda}_{n\rho}^n \xi_1^\rho + \overline{\Lambda}_{nn}^n}{(\overline{\Lambda}_n^n + \overline{\Lambda}_\beta^n \xi_1^\beta)^2} \\ & - \overline{\xi}_1^\alpha \, \frac{\overline{\Lambda}_\rho^n \xi_3^\rho + 3\overline{\Lambda}_{\rho\sigma}^n \xi_2^\rho \xi_1^\sigma + 3\overline{\Lambda}_{n\rho}^n \xi_2^\rho + \overline{\Lambda}_{\rho\sigma\tau}^n \xi_1^\rho \xi_1^\sigma \xi_1^\tau + 3\overline{\Lambda}_{n\rho\sigma}^n \xi_1^\rho \xi_1^\sigma + 6\overline{\Lambda}_{nn\rho}^n \xi_1^\rho + \overline{\Lambda}_{nnn}^n}{(\overline{\Lambda}_n^n + \overline{\Lambda}_\beta^n \xi_1^\beta)^2}\end{aligned}$$

For $k > 1$, the coordinate transformation law is an action of G^k_n. For $k = 1$, however, both the coordinate and parameter transformation laws for first order curve elements are linear. In fact, the parameter transformations have the same form as the transformations of the dilation subgroup of G^1_n.

Let D be the dilation subgroup (a normal subgroup) of G^1_n consisting of those $j^1_0 a \varepsilon G^1_n$ for which $a^i_j = \lambda \delta^i_j$ with $\lambda \neq 0$. Then G^1_n/D, also denoted by PG(n-1), is a factor group, and the associated fiber bundle of the principal bundle $H^1(M)$ corresponding to the dilatation subgroup D turns out to be a principal bundle

$$H^1(M)/D = \langle H^1(M)/D,\ \pi_{H^1/D},\ M,\ G^1_n/D\rangle ,$$
$$= \langle H^1(M)/D,\ \pi_{H^1/D},\ M,\ PG(n-1)\rangle \tag{A5.7}$$

called the <u>bundle of projective n^1-frames</u> or <u>projective linear frames</u>.

The set of all pairs $(p, j^k_p \xi)$ is denoted by $H^k_1(M)/G^k_1 = \mathbb{D}^k_1(M)$, and the projection maps $\pi_{D^k_1}: \mathbb{D}^k_1(M) \to M$ and $\pi^{D^\ell_1}_{D^k_1}: \mathbb{D}^k_1(M) \to \mathbb{D}^\ell_1(M)$ are defined in the usual way.

Set $\mathbb{D}^k_1(\mathbb{R}^n_0) = H^k_1(\mathbb{R}^n_0)/G^k_1$. Then for $k > 1$ the associated bundle of k-directions is the structure

$$\mathcal{H}_1^k(M)/G_1^k = \langle H_1^k(M)/G_1^k,\ \pi_{H_1^k/G_1^k},\ M,\ H_1^k(\mathbb{R}_0^n)/G_1^k,\ \mathcal{H}^k(M)\rangle \tag{A5.8}$$

or

$$\mathcal{D}_1^k(M) = \langle \mathbb{D}_1^k(M),\ \pi_{\mathbb{D}_1^k},\ M,\ \mathbb{D}_1^k(\mathbb{R}_0^n),\ \mathcal{H}^k(M)\rangle.$$

The associated bundle of one-directions is the structure

$$\mathcal{H}_1^1(M)/G_1^1 = \langle H_1^1(M)/G_1^1,\ \pi_{H_1^1/G_1^1},\ M,\ H_1^1(\mathbb{R}_0^n)/G_1^1,\ \mathcal{H}^1(M)/D\rangle \tag{A5.9}$$

or

$$\mathcal{D}_1^1(M) = \langle \mathbb{D}_1^1(M),\ \pi_{\mathbb{D}_1^1},\ M,\ \mathbb{D}_1^1(\mathbb{R}_0^n),\ \mathcal{H}^1(M)/D\rangle.$$

These structures are necessary for a coordinate free discussion of the equations of motion of material bodies.

6. DIRECTING FIELDS

Of particular importance for a coordinate free discussion of the equations of motion of material bodies is the notion of a set of distinct directing field structures each of whose members corresponds to a distinct path structure on M, and the notion of a subset of such structures called geodesic directing fields, each of whose members corresponds to a distinct projective structure on M.

DEFINITION A6.1: A directing field on M is a map $\Xi: \mathbb{D}^1_1(M) \to \mathbb{D}^2_1(M)$ such that $\pi^{\mathbb{D}^1_1}_{\mathbb{D}^2_1} \circ \Xi$ is the identity map on $\mathbb{D}^1_1(M)$, where $\pi^{\mathbb{D}^1_1}_{\mathbb{D}^2_1}: \mathbb{D}^2_1(M) \to \mathbb{D}^1_1(M)$ is the natural projection.

A directing field Ξ defines a second order differential equation for paths on M. In terms of local coordinates $(x^i(p), \xi^\alpha_1)$ for $\mathbb{D}^1_1(M)$ and $(x^i(p), \xi^\alpha_1, \xi^\alpha_2)$ for $\mathbb{D}^2_1(M)$, the map Ξ is given by

$$\Xi\,(x^i(p),\ \xi^\alpha_1) = (x^i(p),\ \xi^\alpha_1,\ \Xi^\alpha_2(x^i(p),\ \xi^\alpha_1)) \qquad \text{(A6.1)}$$

where the $\Xi^\alpha_2(x^i(p), \xi^\alpha_1)$ are (n-1) functions of the (2n-1) variables $(x^i(p), \xi^\alpha_1)$.

The path ξ on M is a solution to the differential equation defined by Ξ, if and only if

$$\Xi \circ j^1\xi = j^2\xi \qquad \text{(A6.2)}$$

which in terms of local coordinates may be written as

$$\frac{d^2x^\alpha}{(dx^n)^2}(x^n) = \Xi_2^\alpha\left(x^n,\ x^\alpha(x^n),\ \frac{dx^\alpha}{dx^n}(x^n)\right). \tag{A6.3}$$

By the fundamental existence theorem, if the direction is specified at some event, that is, if an element $(x_0^n,\ x^\alpha(x_0^n), \frac{dx^\alpha}{dx^n}(x_0^n)) \varepsilon \mathbb{D}_1^1(M)$ is specified for some initial value x_0^n, then $\Xi \circ j^1\xi = j^2\xi$ determines a unique path ξ in M. Moreover, the path $j^1\xi$ in $\mathbb{D}_1^1(M)$ consisting of the points $(x^n,\ x^\alpha(x^n),\ \frac{dx^\alpha}{dx^n}(x^n))$, where $x^\alpha(x^n)$ is the solution of (A6.3) for the given initial data, does not self intersect. $\Xi_*(\mathbb{D}_1^1(M))$ is a (2n-1) dimensional submanifold of the (3n-2) dimensional manifold $\mathbb{D}_1^2(M)$ and the lifts $j^2\xi$ of the solution paths ξ are paths in the (2n-1) dimensional manifold.

The family of all paths ξ corresponding to a given directing field Ξ on M is called a <u>path structure</u> on M.

An important class of directing fields are the geodesic directing fields. For emphasis, Π rather than Ξ will be used to denote a <u>geodesic</u> directing field.

DEFINITION A6.2: A geodesic directing field is a directing field Π: $\mathbb{D}_1^1(M) \to \mathbb{D}_1^2(M)$ for which, at each $p\varepsilon M$, there exists some local coordinates $(\bar{x}^i(p),\ \bar{\xi}_1^\alpha,\ \bar{\xi}_2^\alpha)$ corresponding to a chart $(\bar{U},\bar{x})_p$, such that

$$\bar{\Pi}(\bar{x}^i(p),\ \bar{\xi}_1^\alpha) = (\bar{x}^i(p),\ \bar{\xi}_1^\alpha,\ 0) \tag{A6.4}$$

or

$$\bar{\Pi}_2^{\alpha}(\bar{x}^i(p),\ \bar{\xi}_1^{\alpha}) = 0. \qquad \text{(A6.5)}$$

Note that every geodesic directing field corresponds to an equivalence class of projectively equivalent symmetric linear connections.

THEOREM A6.1: A directing field Π is geodesic if and only if to <u>any</u> local system of coordinates $(x^i(p);\ \xi_1^{\alpha},\ \xi_2^{\alpha})$

$$\Pi_2^{\alpha}(x^i(p),\xi_1^{\alpha}) = \xi_1^{\alpha}[\Pi_{\rho\sigma}^{n}(x^i(p))\xi_1^{\rho}\xi_1^{\sigma}+2\Pi_{n\rho}^{n}(x^i(p))\xi_1^{\rho}+\Pi_{nn}^{n}(x^i p))]$$
$$- [\Pi_{\rho\sigma}^{\alpha}(x^i(p))\xi_1^{\rho}\xi_1^{\sigma}+2\Pi_{n\rho}^{\alpha}(x^i(p))\xi_1^{\rho}+\Pi_{nn}^{\alpha}(x^i(p))]. \qquad \text{(A6.6)}$$

Since $\Pi_{ji}^{i}(x^i(p)) = 0$, $\Pi_{n\rho}^{n}(x^i(p))$ and $\Pi_{nn}^{n}(x^i(p))$ may be eliminated from (A6.6). The expression (A6.6) follows from the transformation law (A5.6).

It is clear from Theorem A6.1, that if Π is a geodesic directing field, then $\Pi_2^{\alpha}(x^i(p),\ \xi_1^{\alpha})$ is a cubic polynomial in ξ_1^{α} in <u>every</u> coordinate chart $(U,x)_p$. The converse is also true.

THEOREM A6.2: If with respect to <u>any</u> coordinate chart $(U,x)_p$, the corresponding map $\Xi_2^{\alpha}(x^i(p),\ \xi_1^{\alpha})$, which determines the directing field Ξ, is cubic, that is, if

$$\Xi_2^{\alpha}(x^i(p),\xi_1^{\alpha}) = A^{\alpha} + B_{\rho}^{\alpha}\xi_1^{\rho} + C_{\rho\sigma}^{\alpha}\xi_1^{\rho}\xi_1^{\sigma} + D_{\rho\sigma\tau}^{\alpha}\xi_1^{\rho}\xi_1^{\sigma}\xi_1^{\tau} \qquad \text{(A6.7)}$$

where the coefficients A, B, C, D are functions only of $p \varepsilon M$, then Ξ is geodesic.

The proofs of the above theorems will not be reproduced here and may be found in [1].

7. THE AFFINE AND PROJECTIVE G-STRUCTURE

The description of geodesic acceleration fields and geodesic directing fields in the previous section concentrated on the role of these geometric object fields as differential equations. Within the context of spacetime such fields appear as equations of motion for physical systems.

There exists a different way of characterising these types of geometric object fields which emphasises their role as fields that determine the geometric structure of the manifold M. When their role as geometric structural fields is of significance, then it is more appropriate to describe them as G-structures rather than as differential equations. The affine and projective structures of spacetime are such geometric structural fields.

Let $P(M)$ be a principal bundle

$$P(M) = \langle P(M), \pi_P, M, G \rangle \tag{A7.1}$$

and let H be a closed subgroup of the structural group G of $P(M)$. An H-structure on M is a reduction to the subgroup of G. That is, an H-structure on an n-dimensional manifold M is represented by the subbundle of $P(M)$, the <u>reduced principal bundle $H(M)$</u> with structure group H. Such an H-structure is in bijective correspondence with cross sections of the <u>associated fiber bundle</u>

$$P(M)/H = \langle P(M)/H,\ \pi_{P/H},\ M,\ G/H,\ P(M)\rangle \qquad (A7.2)$$

of orbits under the action of H of elements of P(M). Hence we shall also say that an H-structure on M is given by a global cross section $\sigma: M \to P(M)/H$ of (A7.2) which specifies an equivalence class of H-related elements of $P(M_p)$, for all $p \varepsilon M$.

Since equivalence classes are difficult to work with a cross section $\sigma: M \to P(M)/H$ of (A7.2) is represented by a family of local cross sections $\sigma_U: U \to P(U)$, such that for all $p \varepsilon U$, $\sigma_U(p) \varepsilon \sigma(p)$. If σ_{U_1} and σ_{U_2} are any two such local cross sections, such that $U_1 \cap U_2 \neq \emptyset$, then there exists a local cross section

$$\rho_{12}: U_1 \cap U_2 \to (U_1 \cap U_2) \times H,$$

such that for all $p \varepsilon U_1 \cap U_2$,

$$\sigma_{U_2}(p) = \sigma_{U_2}(p) \circ \rho_{12}(p). \qquad (A7.3)$$

The transformation (A7.3) is a local gauge transformation which is an active transformation and not a coordinate transformation.

FIRST ORDER G-STRUCTURES

Consider the principal bundle $H^1(M)$ of n^1-frames.

For any closed Lie subgroup SG_n^1 of G_n^1, a cross section of the associated fiber bundle

$$H^1(M)/SG_n^1 = \langle H^1(M)/SG_n^1, \pi_{H^1/SG_n^1}, M, G_n^1/SG_n^1, H^1(M)\rangle \quad (A7.4)$$

defines an SG_n^1-structure, that is, a field of equivalence classes of SG_n^1-related n^1-frames.

If $SG_n^1 = G_n^1$, then each fiber $H^1(M_p)/SG_n^1$ contains only one equivalence class and there is a unique cross section of $H^1(M)/SG_n^1$. In this degenerate case, the cross section does not provide M with any structure at all.

The other extreme occurs when SG_n^1 contains only the identity element of G_n^1. The associated fiber bundle $H^1(M)/SG_n^1$ is then just $H^1(M)$ itself, since each frame is an equivalence class. In this case, a cross section $g: M \to H^1(M)/SG_n^1$ selects a unique frame at each point of the manifold. If $H^1(M)$ admits such a cross section, then M is said to be parallelisable.

Between these extremes lie cases of considerable interest. A Riemannian structure is defined by a cross section of the bundle of equivalence classes of O_n^1-related n^1-frames

$$H^1(M)/O_n^1 = \langle H^1(M)/O_n^1, \pi_{H^1/O_n^1}, M, G_n^1/O_n^1, H^1(M)\rangle. \quad (A7.5)$$

It can be shown that a cross section of the associated fiber bundle $H^1(M)/O_n^1$ exists for any manifold.

On the other hand, not every manifold admits a <u>Lorentz</u> or <u>pseudo-Riemannian structure</u> defined by a cross section of the bundle of equivalence classes of $O^1_{1,n-1}$-related frames (Lorentz-related frames)

$$H^1(M)/O^1_{1,n-1} = \langle H^1(M)/O^1_{1,n-1}, \pi_H{}^1/O^1_{1,n-1}, M, G^1_n/O^1_{1,n-1}, H^1_0(M)\rangle. \tag{A7.6}$$

An $O^1_{1,n-1}$-structure on M is a (global) cross section of $H^1(M)/O^1_{1,n-1}$. Such a cross section may be locally represented by a family of local cross sections of $H^1(M)$. Any two such local cross sections $\sigma_U: U \to H^1(U)$ and $\bar{\sigma}_U: U \to H^1(\bar{U})$ are related by a <u>right acting local $O^1_{1,n-1}$ gauge transformation</u>, defined by a cross section

$$R: U \to U\times O^1_{1,n-1}. \tag{A7.7}$$

Another example of a first order geometric structure is the <u>conformal structure</u>. A first order conformal or $C^1_{1,n-1}$ structure on M is a reduction of the structure group G^1_n of the bundle of n^1-frames $H^1(M)$ to the subgroup $C^1_{1,n-1}$. That is, a $C^1_{1,n-1}$ structure is defined by a cross section of the bundle of equivalence classes of $C^1_{1,n-1}$-related n^1-frames

$$H^1(M)/C^1_{1,n-1} = \langle H^1(M)/C^1_{1,n-1}, \pi_H{}^1/C^1_{1,n-1}, M, G^1_n/C^1_{1,n-1}, H^1(M)\rangle.$$

(A7.8)

As before, any two cross sections of the family of local cross sections σ_U: $U \to H^1(U)$, representing the cross section σ: $M \to H^1(M)/C^1_{1,n-1}$, are related by a right acting local $C^1_{1,n-1}$ gauge transformation defined by the cross section

$$R: U \to U \times C^1_{1,n-1},$$

such that for all $p \varepsilon U$,

$$\bar{\sigma}_U(p) = {}_U(p) \circ R(p)$$

where $R(p) \varepsilon C^1_{1,n-1}$.

A first order conformal or $C^1_{1,n-1}$-structure provides at each point M an infinitesimal light cone. Since this structure distinguishes among future timelike, past timelike, future lightlike, past lightlike and space directions, it determines the causal structure of space-time.

HIGHER ORDER G-STRUCTURES

Geometric structures of higher order may be defined in a similar way. If SG^k_n is a closed Lie subgroup of

G_n^k, then an SG_n^k-structure on a manifold M is given by a cross section of the associated fiber bundle $H^k(M)/SG_n^k$. Such a cross section specifies at each point $p\varepsilon M$ an equivalence class of SG_n^k-related n^k-frames.

It is customary to describe the affine and projective structures of spacetime as connections on the principal bundle $H^1(M)$ of n^1-frames. These structures will however be presented here as <u>second order G-structures</u>. That is, the affine and projective structures will be defined respectively by a reduction to the affine subgroup Γ_n^2 and the projective subgroup P_n^2 of the structure group G_n^2 of the principal bundle of n^2-frames.

$$H^2(M) = \langle H^2(M), \pi_{H^2}, M, G_n^2 \rangle. \qquad \text{(A7.9)}$$

Recall that the set of diffeomorphisms $a: \mathbb{R}^n \to \mathbb{R}^n$, which satisfies $a(\vec{0}) = \vec{0}$, form a group under map composition $c = a \circ b$, and that the k-jet, $j_0^k a$, of such maps, form a Lie group with the group product given by

$$j_0^k c = j_0^k a \circ j_0^k b.$$

The subgroup of G_n^2, whose elements are of the form $(a_j^i, 0)$, consists of the elements of 2 jets, $j_0^2 a$, for which $a_{jk}^i = 0$. This subgroup is denoted by Γ_n^2 and Γ_n^2 is isomorphic to G_n^1. The restriction of the right, free action G_n^2 on $H^2(M)$ to the subgroup Γ_n^2, may be used to construct the associated bundle of equivalence classes of Γ_n^2-related n^2-frames

$$H^2(M)/\Gamma_n^2 = \langle H^2(M)/\Gamma_n^2, \pi_{H^2/\Gamma_n^2}, M, G_n^2/\Gamma_n^2, H^2(M)\rangle, \quad (A7.10)$$

where the <u>typical fiber</u> G_n^2/Γ_n^2 is the space of <u>right</u> cosets.

For each $p\varepsilon M$ define an equivalence relation on $H^2(M_p)$ as follows: any two n^2-frames, $j_0^2\hat{h}$, $j_0^2h\varepsilon H^2(M_p)$, are Γ_n^2-equivalent if and only if there exist $j_0^2a\varepsilon\Gamma_n^2$, such that

$$j_0^2\hat{h} = j_0^2h\circ j_0^2a. \quad (A7.11)$$

The element $j_0^2h\Gamma_n^2\varepsilon H(M)/\Gamma_n^2$, determined by j_0^2h, is then the equivalence class of n^2-frames related to j_0^2h by an element $j_0^2a\varepsilon\Gamma_n^2$; that is,

$$\begin{aligned} j_0^2h\Gamma_n^2 &= \{j_0^2h\circ j_0^2a \,|\, j_0^2a\varepsilon\Gamma_n^2\} \\ &= \{(h_r^i a_j^r, h_{rs}^i a_j^r a_k^s) \,|\, (a_j^i, 0)\varepsilon\Gamma_n^2\}. \end{aligned} \quad (A7.12)$$

The <u>affine structure</u> on M is a cross section $\Gamma\colon M \to H^2(M)/\Gamma_n^2$ of the associated fiber bundle $H^2(M)/\Gamma_n^2$. For each $p\varepsilon M$, $\Gamma(p)$ is an equivalence class of n^2-frames. Such a cross section may be represented by a family of local cross sections $\sigma_U\colon U \to H^2(U)$ of the principal bundle $H^2(M)$, such that for all $p\varepsilon U$, $\sigma_U(p)\varepsilon\Gamma(p)$. These cross sections are determined up to a right acting Γ_n^2 gauge transformation defined by the cross section

$$R\colon U \to U\times\Gamma^2_n. \tag{A7.13}$$

Hence if $\bar{\sigma}_U$, $\sigma_U \varepsilon \Gamma(p)$, then $\bar{\sigma}_U(p) = \sigma(p)\circ R(p)$, where $R(p)\varepsilon\Gamma^2_n$.

For a given computation it is often convenient to work with a particular cross section $\sigma_U\colon U \to H^2(M)$. Such a representative is selected by imposing a coordinate dependent gauge fixing condition, which amounts to choosing a particular set of coordinates for the right coset space G^2_n/Γ^2_n. Imposing the coordinate chart dependent gauge fixing condition

$$h^i_r a^r_j = \delta^i_j, \tag{A7.14}$$

the standard representative for the equivalence class is

$$(\delta^i_j, -\Gamma^i_{jk}), \tag{A7.15}$$

where $\Gamma^i_{jk} = -h^i_{rs} h^{-1r}_{\ \ j} h^{-1s}_{\ \ k}$.

Under a coordinate transformation the standard n^2-frame $(\delta^i_j, -\Gamma^i_{jk})$ is transformed on the left by $(\bar{\Lambda}^i_j, \bar{\Lambda}^i_{jk})\varepsilon G^2_n$ and is returned to standard form on the right by $(\bar{\Lambda}^{-1i}_{\ \ j}, 0)$. Thus

$$(\delta^i_j, -\bar{\Gamma}^i_{jk}) = (\bar{\Lambda}^i_j, \bar{\Lambda}^i_{jk})(\delta^i_j, -\Gamma^i_{jk})(\bar{\Lambda}^{-1}{}^i_j, 0)$$

(A7.16)

$$= (\bar{\Lambda}^i_j, -\bar{\Lambda}^i_r \Gamma^r_{jk} + \bar{\Lambda}^i_{jk})(\bar{\Lambda}^{-1}{}^i_j, 0)$$

$$= (\delta^i_j, -\bar{\Lambda}^i_r \Gamma^r_{st} \bar{\Lambda}^{-1}{}^s_j \bar{\Lambda}^{-1}{}^t_k + \bar{\Lambda}^i_{st} \bar{\Lambda}^{-1}{}^s_j \bar{\Lambda}^{-1}{}^t_k)$$

from which follows the usual transformation law

$$\bar{\Gamma}^i_{jk} = \bar{\Lambda}^i_r \bar{\Lambda}^{-1}{}^s_j \bar{\Lambda}^{-1}{}^t_k \Gamma^r_{st} - \bar{\Lambda}^i_{st} \bar{\Lambda}^{-1}{}^s_j \bar{\Lambda}^{-1}{}^t_k . \qquad \text{(A7.17)}$$

It is clear that the structure group (or microcovariance group) is G^2_n.

The diagram below may prove helpful in illustrating some of the ideas discussed above.

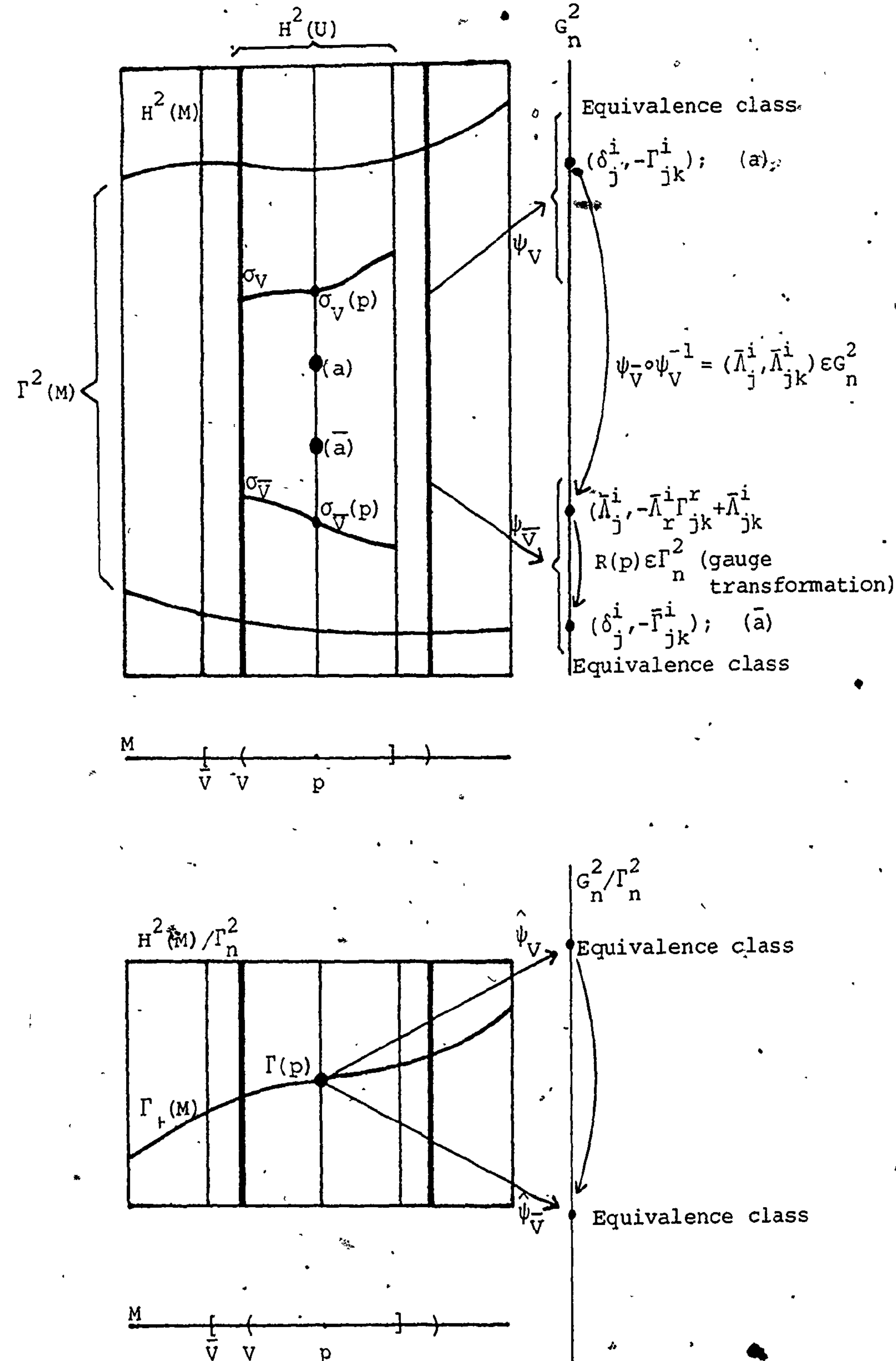

H²(U)
G²ₙ
H²(M)
Equivalence class
(δⁱⱼ, −Γⁱⱼₖ); (a)
ψ_V
σ_V
σ_V(p)
(a)
Γ²(M)
ψ_V̄ ∘ ψ_V⁻¹ = (Λ̄ⁱⱼ, Λ̄ⁱⱼₖ) ε G²ₙ
(ā)
σ_V̄
σ_V̄(p)
ψ_V̄
(Λ̄ⁱⱼ, −Λ̄ⁱᵣΓʳⱼₖ + Λ̄ⁱⱼₖ)
R(p) ε Γ²ₙ (gauge transformation)
(δⁱⱼ, −Γ̄ⁱⱼₖ); (ā)
Equivalence class
M
V̄ V p
G²ₙ/Γ²ₙ
H²(M)/Γ²ₙ
ψ̂_V
Equivalence class
Γ(p)
Γ(M)
ψ̂_V̄
Equivalence class
M
V̄ V p

FIGURE 5A

The projective structure may be defined in a similar way. The subgroup of G_n^2 whose elements are of the form $(a_j^i, a_j^i a_k + a_k^i a_j)$ is the projective group P_n^2, which consists of the elements of 2 jets, $j_0^2 a$, for which

$$a_{jk}^i = a_j^i a_k + a_k^i a_j. \tag{A7.18}$$

The restriction of the right, free action of G_n^2 on $H^2(M)$ to the subgroup P_n^2 may be used to construct the associated bundle of equivalence classes of P_n^2-related n^2-frames

$$H^2(M)/P_n^2 = \langle H^2(M)/P_n^2, \pi_{H^2/P_n^2}, M, G_n^2/P_n^2, H^2(M) \rangle, \tag{A7.19}$$

where the typical fiber G_n^2/P_n^2 is the set of right cosets.

For each $p\varepsilon M$ define an equivalence relation on $H^2(M_p)$ as follows: any two n^2-frames, $j_0^2\hat{h}$, $j_0^2 h\varepsilon H^2(M_p)$, are P_n^2 equivalent if and only if there exists $j_0^2 a\varepsilon P_n^2$, such that

$$j_0^2\hat{h} = j_0^2 h \circ j_0^2 a. \tag{A7.20}$$

Then the element $j_0^2 h P_n^2 \varepsilon H^2(M)/P_n^2$ determined by $j_0^2 h$ is the equivalence class of n^2-frames related to $j_0^2 h$ by an element $j_0^2 a\varepsilon P_n^2$, that is,

$$j_0^2hP_n^2 = \{j_0^2h\circ j_0^2a \mid j_0^2a\varepsilon P_n^2\} \tag{A7.21}$$

$$= \{(h^i_j, h^i_{jk})\circ(a^i_j, a^i_j a_k + a^i_k a_j) \mid (a^i_j, a^i_j a_k + a^i_k a_j)\varepsilon P_n^2\}.$$

The projective structure on M is a cross section

$$\Pi\colon\ M \to H^2(M)/P_n^2$$

of the associated fiber bundle $H^2(M)/P_n^2$ of equivalence classes. For each $p\varepsilon M$, $\Pi(p)$ is an equivalence class of P_n^2-related n^2-frames. As before, such cross sections may be represented by a family of local cross sections $\sigma_U\colon U \to H^2(U)$ of the principal bundle $H^2(M)$, such that $\sigma_U(p)\varepsilon\Pi(p)$, for all $p\varepsilon U\subset M$. These local cross sections are determined up to a right acting P_n^2 gauge transformation defined by the cross section

$$R\colon\ U \to U\times P_n^2.$$

If $\bar{\sigma}_U$, $\sigma_U\varepsilon\Pi(p)$, then

$$\bar{\sigma}_U(p) = \sigma_U(p)\circ R(p),$$

where $R(p)\varepsilon P_n^2$.

A standard representative for the equivalence class of P_n^2-related n^2-frames may be defined by first imposing the coordinate chart dependent gauge fixing condition $h^i_r a^r_j = h^i_j$, to obtain

$$(\delta^i_j, \delta^i_j a_k + \delta^i_k a_j - \Gamma^i_{jk}), \tag{A7.22}$$

and then choosing the a_i so that the second element of (A7.22) is traceless, to obtain

$$(n+1)a_i - \Gamma_i = 0, \tag{A7.23}$$

where Γ_i denotes the trace of Γ^i_{jk}. By substituting (A7.23) into (A7.22), the standard representative becomes

$$(\delta^i_j, -\Pi^i_{jk}), \tag{A7.24}$$

where

$$\Pi^i_{jk} \equiv \Gamma^i_{jk} - \frac{1}{(n+1)} (\delta^i_j \Gamma_k + \delta^i_k \Gamma_j). \tag{A7.25}$$

The transformation law for the Π^i_{jk} may be obtained by combining (A7.17) and (A7.25).

8. MICRO SYMMETRIES

The affine and projective structures both have G_n^2 as their microcovariance group. Their microsymmetry groups, however, are respectively isomorphic to Γ_n^2 and P_n^2.

An active local p-transformation (that is, ϕ: $U \to U$, and $\phi(p) = p$) is represented with respect to some coordinate chart (U,x) by the map $\Phi = x \circ \phi \circ x^{-1}$, and the microtransformation is determined by the partial derivatives (Φ^i_j, Φ^i_{jk}) of Φ at $x(p)$.

The transformed affine structure Γ^ϕ is in general different from Γ. The coordinate representatives are related by

$$\Gamma^{\phi i}_{jk} = \Phi^i_r \Gamma^r_{st} \Phi^{-1s}_j \Phi^{-1t}_k - \Phi^i_{st} \Phi^{-1s}_j \Phi^{-1t}_k. \tag{A8.1}$$

The invariance condition is

$$\Gamma^\phi = \Gamma. \tag{A8.2}$$

Hence, the expression (A8.1) may be used to solve for the Φ^i_{jk} in terms of the Φ^i_j and the known Γ^i_{jk}. Consequently, the microsymmetry group of the affine structure Γ at $p \varepsilon M$ is the set of microtransformations of the form

$$(\Phi^i_j,\ \Phi^i_r \Gamma^r_{jk} - \Gamma^i_{st} \Phi^s_j \Phi^t_k) \tag{A8.3}$$

which constitutes a group isomorphic to Γ^2_n.

By a similar but algebraically more involved calculation, it may be shown that the microsymmetry group of a projective structure Π at $p\varepsilon M$ is the set of microtransformations of the form

$$(\phi^i_j,\ \phi^i_r\Pi^r_{jk} - \Pi^i_{st}\phi^s_j\phi^t_k + \phi^i_j\phi_k + \phi^i_k\phi_j) \tag{A8.4}$$

where

$$\phi_i = \frac{1}{n+1}\,\phi^r_{ri}. \tag{A8.5}$$

The standard affine frame at $p\varepsilon M$ with respect to a given local chart (U,x) is $(\delta^i_j,\ -\Gamma^i_{jk})$. The affine frame related to this one by $j^2_0 a\varepsilon\Gamma^2_n$, is

$$(a^i_j,\ -\Gamma^i_{rs}a^r_j a^s_k). \tag{A8.6}$$

Similarly, the projective frame at $p\varepsilon M$ related to the standard projective frame $(\delta^i_j,\ -\Pi^i_{jk})$ by $j^2_0 a\varepsilon P^2_n$ is

$$(a^i_j,\ -\Pi^i_{rs}a^r_j a^s_k + a^i_j a_k + a^i_k a_j). \tag{A8.7}$$

If $c\colon \mathbb{R} \to \mathbb{R}^n$ is a curve with $c(0)=\vec{0}$, then this curve is straight to second order at $0\varepsilon\mathbb{R}$, if and only if

$$j^2_0 c = (c^i_1, 0) \tag{A8.8}$$

The composition $\gamma = h \circ c$: $\mathbb{R} \to M$ is a curve in M through $p \in M$. The image of the second order straight curve (A8.8) under one of the affine frames (A8.6) is the second order curve element

$$j_0^2\gamma = j_0^2 h \circ j_0^2 c \tag{A8.9}$$

$$= (a^i_j c^j_1,\ -\Gamma^i_{rs} a^r_j a^s_k c^j_1 c^k_1)$$

$$= (\gamma^i_1,\ -\Gamma^i_{jk}\gamma^j_1\gamma^k_1)$$

where $\gamma^i_1 \equiv a^i_j c^j_1$. Thus, a second order curve element is affine if and only if

$$\gamma^i_2 + \Gamma^i_{jk}\gamma^j_1\gamma^k_1 = 0. \tag{A8.10}$$

Similarly, the image of (A8.8) under one of the projective frames (A8.7) is

$$j_0^2\gamma = j_0^2 \circ j_0^2 c \tag{A8.11}$$

$$= (a^i_j c^j_1,\ -\Pi^i_{rs} a^r_j a^s_k c^j_1 c^k_1 + 2a^i_j c^j_1 a_k c^k_1)$$

$$= (\gamma^i_1,\ -\Pi^i_{jk}\gamma^j_1\gamma^k_1 + 2a_k a^{-1k}_{\ \ j}\gamma^j_1\gamma^i_1).$$

Since the a_k are arbitrary, a second order curve element is projective if and only if

$$\gamma_2^i + \Pi^i_{jk}\gamma_1^j\gamma_1^k = \lambda\gamma_1^i, \tag{A8.12}$$

where $\lambda = a_k a^{-1k}_{\ \ j}\gamma_1^j$ is arbitrary.

The differential equations for curves which are everywhere affine or projective may be obtained from (A8.10) and (A8.12) by replacing $x^i(p)$ with $\gamma^i(t)$, γ_1^i with $\dot{\gamma}^i(t)$ and γ_2^i with $\ddot{\gamma}^i(t)$, where the dots denote the operator d/dt. In the projective case, a family of equations is obtained since λ may vary smoothly along the curve. The differential equations for affine and projective curves are

$$\ddot{\gamma}^i(t) + \Gamma^i_{jk}(\gamma^i(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0 \tag{A8.13}$$

$$\ddot{\gamma}^i(t) + \Pi^i_{jk}(\gamma^i(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = \lambda(t)\dot{\gamma}^i(t). \tag{A8.14}$$

The differential equations (A8.13) and (A8.14) are determined respectively by an affine and projective structure on M.

The definition of an active transformation of acceleration and directing fields involves the notion of a <u>bundle automorphism</u> (Definition A1.1). For any diffeomorphism ϕ: M → M, the <u>k-prolongation</u>

$$j^k\phi: H_1^k(M) \to H_1^k(M)$$

is the diffeomorphism defined by

$$j_0^k\tilde{\gamma} = j_p^k\phi\circ j_0^k\gamma \tag{A8.15}$$

where $\tilde{\gamma} = \phi\circ\gamma$. For acceleration fields, the situation (slightly more complicated than in (A1.33)) is summarized by the following commutative diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
H_1^2(M) \arrow[r, shift left, "j^2\phi"] \arrow[d, shift right, "\pi^{H_1^1}_{H_1^2}"'] \arrow[dd, bend right=60, "\pi_{H_1^2}"'] & H_1^2(M) \arrow[l, shift left, "j^2\phi^{-1}"] \arrow[d, shift left, "\pi^{H_1^1}_{H_1^2}"] \arrow[dd, bend left=60, "\pi_{H_1^2}"] \\
H_1^1(M) \arrow[u, shift right, "A"'] \arrow[r, shift left, "j^1\phi"] \arrow[d, "\pi_{H_1^1}"'] & H_1^1(M) \arrow[u, shift left, "A^\phi"] \arrow[l, shift left, "j^1\phi^{-1}"] \arrow[d, "\pi_{H_1^1}"] \\
M \arrow[r, shift left, "\phi"] & M \arrow[l, shift left, "\phi^{-1}"]
\end{tikzcd} \tag{A8.16}$$

where

$$A^{\phi} = j^2\phi \circ A \circ j^1\phi^{-1}. \tag{A8.17}$$

The transformation (A8.15) commutes with the parameter transformation $\gamma \circ \mu_t(\mathbb{R}) = \gamma_t(\mathbb{R})$ since the actions are on the left and right respectively. Consequently, the transformation (A8.15) induces the diffeomorphism

$$j^k\phi : \mathbb{D}^k_1(M) \to \mathbb{D}^k_1(M). \tag{A8.18}$$

The pointwise action on k-directions will be indicated by

$$j^k_{\phi(p)}\tilde{\xi} = j^k_p\phi \circ j^k_p\xi. \tag{A8.19}$$

The commutative diagram which summarises the relationship of the bundle automorphisms for the active transformation of a directing field Ξ, namely

$$\Xi^{\phi} = j^2\phi \circ \Xi \circ j^1\phi^{-1} \tag{A8.20}$$

is similar to (A8.16) with the symbols H and A replaced by the symbols $\mathbb{D}$ and Ξ respectively.

The invariance conditions for accelerating and directing fields are

$$A^{\phi} = A \tag{A8.21}$$

$$\Xi^{\phi} = \Xi.$$

THEOREM A8.1: An acceleration field A is geodesic if and only if for all $p \in M$, the microsymmetry at p is isomorphic to Γ_n^2.

Although sufficiency is difficult to prove, necessity follows readily from the fact that under an active p-transformation with $j_p^2\phi = (\Phi_j^i, \Phi_{jk}^i)$,

$$\tilde{\gamma}_2^i = \Phi_j^i\gamma_2^j + \Phi_{jk}^i\gamma_1^j\gamma_1^k. \tag{A8.22}$$

Suppose that the coordinate system has been chosen so that $A_2^i(x^i(p), \gamma_1^i)$ vanishes. Then invariance $(A^{\phi} = A)$ requires that $\tilde{A}_2^i(x^i(p), \tilde{\gamma}_1^i)$ also vanishes. Hence $\gamma_2^i = \tilde{\gamma}_2^i = 0$ in (A8.22). Therefore, $\Phi_{jk}^i = 0$, while Φ_j^i may be chosen arbitrarily.

THEOREM A8.2: A directing field Ξ is geodesic if and only if for all $p \in M$, the microsymmetry group at p is isomorphic to P_n^2.

Again, sufficiency is difficult to establish, but necessity follows readily from the fact that under an active transformation, with $j_p^2\phi = (\Phi_j^i, \Phi_{jk}^i)$, $(\tilde{\xi}_1^{\alpha}, \tilde{\xi}_2^{\alpha})$ is related to $(\xi_1^{\alpha}, \xi_2^{\alpha})$ by equations similar to (A5.6) with

$(\bar{\xi}_1^\alpha, \bar{\xi}_2^\alpha)$ replaced by $(\tilde{\xi}_1^\alpha, \tilde{\xi}_2^\alpha)$ and $(\bar{\Lambda}_j^i, \bar{\Lambda}_{jk}^i)$ replaced by (Φ_j^i, Φ_{jk}^i). In the coordinate system in which the functions Ξ_2^α and $\tilde{\Xi}_2^\alpha$ vanish (or $\xi_2^\alpha = \tilde{\xi}_2^\alpha = 0$), the active version of (A5.6) gives

$$(\Phi_{\rho\sigma}^\alpha \xi_1^\rho \xi_1^\sigma + 2\Phi_{n\rho}^\alpha \xi_1^\rho + \Phi_{nn}^\alpha)(\Phi_n^n + \Phi_\beta^n \xi_1^\beta)$$

$$= (\Phi_n^\alpha + \Phi_\beta^\alpha \xi_1^\beta)(\Phi_{\rho\sigma}^n \xi_1^\rho \xi_1^\sigma + 2\Phi_{n\rho}^n \xi_1^\rho + \Phi_{nn}^n). \tag{A8.23}$$

This condition is satisfied by elements of the form $(\Phi_j^i, 0)$. Since

$$(\Phi_j^i, \Phi_{jk}^i) = (\Phi_j^i, 0)(\delta_j^i, \Phi^{-1i}{}_r \Phi_{jk}^r) \tag{A8.24}$$

attention may be restricted to elements of the form $(\delta_j^i, \Phi_{jk}^i)$. Then (A8.23) becomes

$$(\Phi_{\rho\sigma}^\alpha \xi_1^\rho \xi_1^\sigma + 2\Phi_{n\rho}^\alpha \xi_1^\rho + \Phi_{nn}^\alpha)$$

$$= \xi_1^\alpha (\Phi_{\rho\sigma}^n \xi_1^\rho \xi_1^\sigma + 2\Phi_{n\rho}^n \xi_1^\rho + \Phi_{nn}^n). \tag{A8.25}$$

This expression may be written as a polynomial in ξ_1^α which must vanish for all ξ_1^α. Hence, the coefficients of this polynomial must vanish. The conditions obtained are

$$\Phi_{\rho\sigma}^n = 0; \quad \Phi_{nn}^\alpha = 0$$

$$\Phi^{\alpha}_{\rho\sigma} = \delta^{\alpha}_{\rho}\Phi^{n}_{n\rho} + \delta^{\alpha}_{\sigma}\Phi^{n}_{n\rho} \tag{A8.26}$$

$$\Phi^{\alpha}_{n\rho} = \delta^{\alpha}_{\rho}\Phi^{n}_{nn}/2,$$

which may be written

$$\Phi^{i}_{jk} = \delta^{i}_{j}\Phi_{k} + \delta^{i}_{k}\Phi_{j}, \tag{A8.27}$$

where

$$\Phi_{i} = \frac{1}{n+1}\Phi^{j}_{ji} \tag{A8.28}$$

and

$$\Phi^{n}_{n\sigma} = \Phi_{\sigma},$$
$$\Phi^{n}_{nn} = 2\Phi_{n}. \tag{A8.29}$$

Then the general element of the microsymmetry transformation has the form

$$(\Phi^{i}_{j},\ \Phi^{i}_{j}\Phi_{k} + \Phi^{i}_{k}\Phi_{j}). \tag{A8.30}$$

The microsymmetry transformations for geodesic acceleration and directing fields, derived here in an adapted coordinate system, are given by (A8.3) and (A8.4) in an arbitrary coordinate system.

Clearly, the differential equation determined by a geodesic acceleration field by combining (A4.13) and (A4.12) is the same as the differential equation (A8.13) for affine curves, the equation determined by an affine structure. Although it is less obvious, the family of differential equations of projective curves (A8.14) (with arbitrary $\lambda(t)$) determined by a projective structure Π, is equivalent to the equation determined by a geodesic directing field, by combining (A6.6) with (A6.3). This can be shown by using the equation (A8.14) with $i=n$ to eliminate the parameter t in favor of x^n.

Clearly, there is a one to one correspondence between affine structures and geodesic acceleration fields and between projective structures and geodesic directing fields.

As was pointed out in the beginning of Section A7, the description of affine and projective structures as G-structures emphasises their ontological status as fundamental spacetime structures. Their description in terms of geodesic acceleration and geodesic directing fields (induced by the G-structures) on the other hand, serves to highlight their role in governing the motions of physical systems.

FOOTNOTES

1. The theorems presented here may be found in [1]. Theorem A6.1 was first proved in [5] using the usual T(M) and T(T(M)) formalism. It is proved more simply in [1] using the more appropriate jet formalism of C. Ehresmann, "Les Prolongements d'une Variété Differentiable I-V," C. R. Acad. Sci. 223, pp. 598, 777, 1081 (1951); pp. 1002, 1424 (1952). Section A7 is partly based on [2]. In [2] everything is formulated in terms of co-frames. In addition a special prolongation technique is introduced in [2] which is not included here. The full proofs of Theorems A8.1, A8.2 may be found in [1].

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